SINUS Bavaria
Exploring New Paths in Teaching Mathematics and Science
The SINUS Transfer pilot study was sponsored by the Federal Ministry of Education and Research (BMBF) and by the ministries of education of the Länder (federal states) in the Federal Republic of Germany.

The publication prepared at ISB (Staatsinstitut für Schulqualität und Bildungsforschung – State Institute for Quality Schooling and Research Education) was commissioned by the Bavarian State Ministry of Education and Cultural Affairs (Staatsministerium für Unterricht und Kultus – StMUK).

The Federal Ministry of Education and Research (bmbf) has financially supported the English translation of this document.

The translation has been arranged and coordinated by the Chair of Mathematics and Mathematics Education at Bayreuth University: Prof. Dr. Peter Baptist, Dr. Carsten Miller, Dagmar Raab

Editors: Doris Drexel, Harald Haidl, Emmeram Zebhauser, Monika Zebhauser
Editor-in-Chief: Christoph Hammer
Designed by Agentur2 GmbH, Munich
Picture credits: Daniel Biskup, erysipel/Pixelio (p. 78), Frans Stoppelman/Voller Ernst (p. 64), Stockxpert (p. 35), private
Translated by ReinekeTeam, Heidelberg 2010
The German version was published by the Bavarian State Ministry of Education and Cultural Affairs (StMUK)

We very much appreciate the support received from
Dieter Götzl (StMUK)
Thomas Schäfer (StMUK)
Dr. Hans-Werner Thum (ISB)
Friedrich Schrägle (ISB)

Links last updated: December 2010. No liability accepted for the content of any hyper-links provided.

Last updated: December 2010
Exploring New Paths in
Teaching Mathematics and Science

The English translation of this document was financially
supported by the Federal Ministry of Education and Research (bmbf)
Translated by ReinekeTeam, Heidelberg 2010

The translation has been arranged and coordinated by the Chair
of Mathematics and Mathematics Education at Bayreuth University:
Prof. Dr. Peter Baptist, Dr. Carsten Miller, Dagmar Raab
Content

Introduction
The SINUS Transfer program 6
About this booklet 10
Notes on the effectiveness of science teaching, with special reference to biology and chemistry 11

Encouraging Imagination
Imagination as a key to understanding 16
Suggestions for hands-on training of spatial imagination 30

How to Achieve Sustainable Learning Results
From basic knowledge catalogues to basic concepts 40
Developing knowledge at different levels of understanding 51

How to Strengthen Students’ Individual Responsibility
Approaches to learning by dialogue 62
The math diary: Reflecting on personal progress in mathematics 66

Establishing a Range of Methods
From teacher predominance to methodological diversity 72
Tasks encouraging student activity 86
Making physics a hands-on experience 94

Presenting Novel Tasks
Developing skills with novel tasks 98
Using tasks in designing math lessons 110

Experience Gleaned
Outcome of process evaluation 116
Teachers’ progress reports 121
Experiences in two basic secondary schools 123

Outlook: SINUS Bavaria 126
Preface

A quick glance at the daily paper shows that in the media the quality of our schools has become a much-publicized and hotly-debated issue – and rightly so! The quality of teaching and education is the operative factor in getting the best out of our students. Accordingly, I find it highly gratifying that at schools in Bavaria there are so many people actively involved in systematic, diverse and successful efforts to enhance school quality even further. These efforts focus on classroom instruction. Our objective must be to combine the latest ideas on educational methodology with practical classroom experience, thus giving our children access to the best possible education. In mathematics and the sciences, the SINUS programs show us how to go about it. Bavaria has been participating in these nation-wide programs ever since they started up back in 1998. We are so convinced of their outstanding success that we are now continuing with these activities in the framework of the SINUS Bavaria program.

This booklet addresses both the teachers at the 400 Bavarian SINUS schools and those instructors who have not yet come into contact with the program. It offers a multitude of ideas about how to further enhance teaching standards in the long-term perspective. I would be overjoyed if this booklet and the range of ideas it contains should turn out to be a source of stimulation and enrichment for teaching at our schools. So check out these proposals for their inspiration value! Try putting the ideas into practice. Talk to other members of staff about your experiences and make your own contribution to improving the quality of mathematics and science teaching at our Bavarian schools.

I wish you much pleasure and success in implementing the suggestions made in this booklet!

Munich, December 2007

Siegfried Schneider
Then Bavarian State Minister of Education and Cultural Affairs
Ratsvorsitzender der Stiftung Bildungspakt Bayern
In 1997, after the alarming results of the TIMS Study, the German BLK (Bund-Länder-Kommission für Bildungsplanung und Forschungsförderung – Bund-Länder Commission for Educational Planning and Research Promotion) initiated the SINUS pilot study entitled „Enhancing the Effectiveness of Teaching in Mathematics and the Natural Sciences – [„Steigerung der Effizienz des mathematisch-naturwissenschaftlichen Unterrichts“]. Underlying this pilot program was an expertise1 elaborated by a group of experts under the direction of Prof. Dr. Jürgen Baumert of the MPIB (Max Planck Institute for Human Development, Berlin). With reference to issues connected with the learning and teaching of mathematics, the expertise analyzed the deficiencies of traditional classroom teaching. Furthermore, the experts suggested feasible strategies for the improvement of teaching in mathematics and science.

Initially, 180 German schools of all types providing lower and upper secondary education enrolled in BLK’s SINUS pilot study (1998–2003). As it turned out, the content-related strategies defined and proposed in the expertise were well suited to the purpose of triggering a sustainable enhancement if the quality of teaching at all levels. In Bavaria, 6 Hauptschulen (basic general education), 6 Realschulen (lower secondary education, no equivalent in UK, junior high schools US) and 12 Gymnasien (upper secondary educa-

---

1. BLK: Materialien zur Bildungsplanung und Forschungsförderung; Number 60 (Expert Opinion provided in preparation of the pilot study „Enhancing the Effectiveness of Teaching in Mathematics and Natural Sciences“)
tion; grammar schools UK, senior high schools US) enrolled in the pilot program. In order to disseminate the ideas, approaches, and results connected with a small number of SINUS schools, the BLK launched the Sinus Transfer pilot program in 2003 to explore how to transfer the insights gained to as many schools as possible. This study involved a total of over 1,800 schools in Germany and over 400 schools in Bavaria alone.

Since the end of SINUS Transfer in July 2007, the Länder have been responsible for disseminating the strategies underlying it. Bavaria devised a sophisticated advanced teacher training strategy reaching a large number of schools. Furthermore, August 2005 saw the launching of the 5-year SINUS program for primary schools, involving 120 schools in Germany and 20 in Bavaria. By the school year 2007/2008, the number of schools enrolled had doubled to 240 German and 40 Bavarian schools.

Overall Strategy

SINUS relies on competent and experienced teachers at different levels of schooling. Teachers are expected to make their own decisions about their aims and the means they intend to use for improving their classroom teaching. These are not isolated endeavors but the initiation of a process that will lead to sustainable improvement in classroom teaching. The BLK expertise referred to earlier lists individual modules describing important fields of action and offering general guidance.

- **M1:** Developing a new perception of tasks
- **M2:** Working scientifically
- **M3:** Learning from mistakes
- **M4:** Securing basic knowledge – learning at different levels of comprehension
- **M5:** Heightening student awareness of increased skills – cumulative learning
- **M6:** Heightening student awareness of subject boundaries – interdisciplinary approaches
- **M7:** Encouraging girls and boys equally
- **M8:** Devising tasks geared to student cooperation
- **M9:** Strengthening student responsibility for individual progress
- **M10:** Assessing students’ progress: monitoring and feedback
- **M11:** Assuring quality standards in individual schools and developing general standards valid across all types of schools
A comprehensive description of the individual modules can be found in the expertise submitted to the BLK and in the booklet „Enhancing Teaching in Mathematics and Natural Sciences“².

Staff cooperation in networks

Another contributory factor to the success of the SINUS programs is the way they offer teachers a framework for staff cooperation. There is much more to this than casual coordination with other teachers in the staff room or the exchange of questions for tests. Instead, SINUS expects teachers to cooperate in planning lessons and in analyzing their success afterwards. Such cooperation is not limited to in-school processes. It also extends to groups organized across different schools and has frequently been welcomed as highly beneficial. It contributes to the teachers’ satisfaction with their jobs.

Classroom materials

The biggest misunderstanding in connection with the question of how classroom lessons could be improved effectively and sustainably is the assumption that all you need are better classroom materials. Teachers enrolled in the SINUS program are frequently confronted with the following request: “Just give us your materials and then we’ll adopt the SINUS method as well.” There is a major misconception involved here. There is no such thing as a SINUS teaching method. Teaching is either successful or it isn’t. The degree of success does not depend on the materials used.

Excellent materials have been widely available for a very long time. What really matters is attitude and professionalism, a perception of one’s own teaching. Classroom materials are not the crucial factor, but support, teamwork, and high-quality, tried-and-tested didactic strategies. To support ongoing work in the spirit of SINUS and to provide insights for all interested parties, the examples presented in this booklet come with detailed commentaries designed to indicate how they tie in with the convictions and approaches underlying the SINUS program.

² Bavarian State Ministry of Education and Cultural Affairs: Weiterentwicklung des mathematisch-naturwissenschaftlichen Unterrichts – Erfahrungsbericht zum BLK-Programm SINUS in Bayern; Munich 2002 (Enhancing Teaching in Mathematics and Natural Sciences)
Making It Work

Experienced SINUS teachers team up in pairs to look after groups of schools over a lengthy period of time. It is naturally beneficial if most of the teaching staff (or at least the subject-specialized teaching staff) participate in the program. In a school year, the teacher pair usually offers support on four afternoons and organizes one all-day session. The SINUS twosomes use the afternoon meetings to provide didactic and methodological advice, which is then taken up by the teachers in group sessions at which they plan their own classroom lessons. The teaching strategies thus devised are subsequently implemented and put to the test in real school settings. At the next session, the participants report and reflect on the experience they have gathered. Gradually, a wide range of strategies accumulate that are geared to the modules referred to earlier. After discussion, they can be immediately implemented in the classroom. In addition, teachers can broach issues arising from their ongoing teaching experience and/or developments specific to the kind of school they are teaching at. The all-day sessions bring together the teachers from all the schools supported by a given SINUS twosome. Components of these sessions include workshops prepared by the participants and lectures held by renowned didactic experts.

The SINUS coordinators are present at the schools involved and can thus immediately respond to and follow up issues arising in class. They continuously support teachers in exploring novel paths toward better teaching. Voluntary participation is an indispensable prerequisite for the willingness to embark on the process of enhancing one’s own teaching and for the formation of functional networks. Readiness to change one’s own teaching approach cannot be induced by forced participation in ongoing teacher training courses. We have already observed positive effects in other subjects resulting from the SINUS initiative to explore new paths toward enhancing teaching methods. However, there has so far been no systematic effort to capitalize on this development, as one key to the success of the SINUS project is the way it is specifically geared to the relevant subject(s). However, strategies based on a modular approach and geared to professional cooperation in networks have proven to be valid and sustainable, independently of the respective subject matter. Accordingly, it could make good sense to think seriously about how the SINUS approach could be transferred to other subjects.
Among the major challenges facing teachers at present are adjustments to new curricula with greater scope for designing their own classroom teaching, the further development of centralized school-leaving examinations, and response to higher educational standards. All this increases the need for ongoing teacher training. These developments and SINUS are both geared to the significant objective of designing classwork that is oriented to student skills. In future, it will not be enough to inquire what topics teachers have covered in their lessons. The focus will shift toward developing the students’ skills and enhancing their ability to work independently.

About this Booklet

General Information

The individual contributions published in this booklet reflect the existing diversity of the work being done in different school types, school groups, and subjects. There are various authors involved and they all have their own way of expressing themselves. No attempt has been made by the editors to achieve stylistic uniformity. They have opted for diversity, the same diversity the SINUS program stands for.

The Subjects

Although most of the individual contributions are related to specific subjects, many of the ideas and strategies discussed here are equally relevant for other subjects as well. An example taken from a biology lesson may be just as convincing for math teachers. If so, they will then adapt their approach and implement the given strategy in their own classroom teaching. The BLK expertise¹ and the booklet entitled „Weiterentwicklung des mathematisch-naturwissenschaftlichen Unterrichts“² contain comprehensive descriptions of problematic issues encountered in teaching mathematics and natural sciences. Accordingly, we refrain from an explicit description in this booklet. A group of biology and chemistry teachers set out to identify the impediments getting in the way of enhanced teaching effectiveness in these subjects.

¹ BLK: Materialien zur Bildungsplanung und Forschungsförderung; Heft 60 (Gutachten zur Vorbereitung des Programms „Steigerung der Effizienz des mathematisch-naturwissenschaftlichen Unterrichts“) (Expert Opinion provided in preparation of the pilot study „Enhancing the Effectiveness of Teaching in Mathematics and Natural Sciences“)
² Bavarian State Ministry of Education and Cultural Affairs: Weiterentwicklung des mathematisch-naturwissenschaftlichen Unterrichts – Erfahrungsbericht zum BLK-Programm SINUS in Bayern; Munich 2002
The core objectives of education at a Gymnasium are preparing students for university and providing a general education that will remain with them beyond their school years. Bearing in mind that even applicants to degree programs in the sciences are hardly expected to know subject-specific facts as a prerequisite, the essential aim at this level must be to instill in the students basic knowledge and comprehension as well as basic skills and attitudes that will stand them in good stead later. For this reason, effectiveness in teaching does not mean squeezing in as much detailed knowledge as possible within an already short amount of classroom time.

These objectives are also expressed in the national educational standards¹ that have been in place since 2005. According to these objectives, students must acquire certain skills by the end of their 10th school year; these skills must take into account not only the subject matter, but also the inculcation of proactive skills in acquiring, communicating, and evaluating knowledge. The aspects related to the subject matter described as “basic concepts” (see the

¹ www.kmk.org/schul/home1.htm
subject profiles in the curriculum) are represented in the graphic above as “basic knowledge and comprehension”; the action dimension is covered by “basic skills” and “basic attitudes.”

Problems Preventing Effective Teaching in Biology and Chemistry

In order to identify the most relevant problem areas, participants in SINUS Transfer began by addressing the following questions on teaching effectiveness in both subjects:

➔ “What are the biggest impediments to effectiveness?” (yellow cards)

➔ “What are we already doing about them?” (green cards)

In the process, the responses fell into three areas:

Lack of basic knowledge

There was general consensus that the students’ inadequate basic knowledge was a problem. This corresponds to situations that many biology and chemistry teachers have encountered in the course of their careers, such as when upper-level students are unable to draw on basic material they should have learned earlier, or when learners have memorized individual concepts but are unable to place them in a larger context.
Other major impediments listed were the passivity of many students and their poor work habits. They obviously expect their teachers to serve up the material in appealingly packaged little portions that they can easily consume.

The teachers also agreed that effective teaching is often thwarted by unfavorable conditions, such as class size, in combination with space constraints and inadequate materials. Clearly, the low level of importance attributed to biology and chemistry as school subjects by the general public also plays a role.

**Strategies Offered by the SINUS Program**

The first two problem areas are covered by SINUS modules: “Securing basic knowledge – learning at different levels of comprehension”, “Heightening student awareness of increased skills – cumulative learning”, “Strengthening student responsibility for individual progress”, and “Developing a new perception of tasks”.

Experience shows that students often get to know biology and chemistry as an accumulation of bits of knowledge that are learned briefly and then forgotten again. More than ever before, the goal must be to draw their attention to the basics and to ensure that these basics are firmly entrenched. The perception of their own increasing skills is a major motivation factor for students. In addition, their learning must be cumulative, that is, they must understand how the individual items presented to them relate to each other. At the same time, cumulative learning is also the basis for understanding more complex issues.

Genuine teaching improvement largely depends on how successful teachers are in strengthening students’ willingness and ability to assume responsibility for their own progress and to learn effective strategies for it; this includes students acquiring various methods of assimilating and representing information and retaining them as basic skills for a long time. This leads away from teacher-dominated instruction to a variety of different methods.
Tasks play a central role in teaching mathematics and science. Their importance ranges from learning motivation and support in understanding and appropriating new material to practicing, applying, and securing acquired knowledge. Developing a new perception of tasks can do a great deal to improve teaching.

Suggestions for Implementation within the Department

Over the 9 years of the SINUS programs, experience has shown that effective measures require that if possible all of the members of a department be involved in the process, during which departments can enlist support from the SINUS teacher pairs.

Step 1: Suggestions
Gathering and analyzing the problems that interfere with each participant’s teaching can take place at the beginning of the process.

Step 2: Clarification of aims
Next should follow a discussion within the department about goals and the extent to which they can be implemented. It is important at this point to at least reach a basic minimum consensus.

Step 3: Working phase
A working phase follows the discussion; during this phase, the focus should be on a specific task, such as preparing a catalogue of basic knowledge or working out concrete teaching plans.

Step 4: Implementation
The last step is implementing the agreed-on goals during class.

In the main part of this booklet, numerous suggestions will be made with regard to subject matter.
Encouraging Imagination

Imagination as a Key to Understanding

The Meaning of Imagination in Mathematics

“There is knowledge unworthy of the name, a debased and debasing variety that relegates us to the status of operative factors in an automatic process.”

With this statement in an article\(^1\) from 1948, Martin Wagenschein was already pointing to problems deriving from the rote learning of mathematics, in which the subject matter is not sufficiently linked to imagination. This means that students’ mathematical abilities remain limited to the application of rules or formulas, carrying out memorized algorithms for familiar tasks. They are unable to transfer
their knowledge to new types of problem or to use mathematics as a tool for modeling. Imagination does not always have to take the form of an image. In this article, imagination refers to concepts that are generally meaningful, that represent students’ mathematical knowledge. Often, very different ideas are necessary in order to capture the full scope of a topic. Consider the following example, where the fraction has to be linked to different ideas or images:\(^2\)

**Imagining a portion:** Imagining a portion: Maria has \(\frac{3}{4}\) of a pizza.
(a fraction as part of a whole)

![Pie chart](image)

**Imagining an operator:** Mike won \(\frac{3}{4}\) of 120 €.
How much money did he receive? (a fraction as an arithmetical statement or a statement of amount or size)

![Fractions and operations](image)

**Imagining a number:** (a fraction as the description of a number or absolute statement of size)

![Number line](image)

There have been several recent studies\(^3\) looking at students’ imagination. Of particular importance is their agreement on the discovery that errors – particularly those commonly interpreted as being careless mistakes – are commonly caused by deficits in imagination. A survey carried out as part of the PALMA project showed that roughly half of all errors were caused by deficiencies of this kind. Students in year 6 in different types of schools – Hauptschule, Realschule, and Gymnasium – were given the following problem:


\(^3\) For example, PALMA
www.uni-regensburg.de/fakultaeten/nat_Fak_I/BIOQA/
Is there a fraction that is bigger than $\frac{1}{3}$ and smaller than $\frac{1}{2}$?

Only 22% of the participants were able to answer this correctly. In interviews afterward, it became apparent that the reason for the wrong answer was often due to faulty logic. They reasoned that since there is no whole number between 2 and 3, there cannot be a fraction between $\frac{1}{3}$ and $\frac{1}{2}$. For these students, it is obvious that the introduction of fractions was not accompanied by adequate further development of their feeling for numbers. Their training in imagining fractions had been inadequate. This induced them to transfer the characteristics of the more familiar whole numbers to fractions.

The studies referred to above found that there were real problems of imagination preventing students not only from understanding numbers – and fractions in particular – but also in understanding the workings of decimal numbers.

Deficits in these areas are certain to lead to problems with more advanced material. A well-founded understanding of the place-value system for whole numbers is one of the prerequisites for understanding rational numbers in decimal notation, for reckoning with measurements or dimensions, or for being able to round. Incorrect ideas about multiplication, or no ideas at all, have a negative effect on calculating areas, for example, or calculating with terms. If these gaps in understanding are not detected, they will dog the students through their school careers, even into upper-level coursework, where problems will probably not be connected to deficits in mathematical imagination.
Consequences for the Classroom

When working through new material, it is important to know which conceptions students have already formed about the topic. The question is how we can go about checking these conceptions.

One good way to do this is to have students visualize mathematical content. Here is an example from a class of students in the 6th year at a Realschule; they were given the following task:

Draw a picture for each of the terms, without using numbers or mathematical symbols:

\[ 2 + 3; \ 3 \cdot 4; \ 6 - 4; \ 15 : 3. \]

For the most part, students did a fine job of depicting addition and subtraction in picture form, but they had problems visualizing multiplication and division. Very few students were able to draw appropriate pictures of them.

**Addition and subtraction:**

The student divided by 5 instead of 3.

**Division:**

The correct answer.
From this we can see that students are not doing an adequate job of thinking imaginatively about multiplication and division.

In many cases, students can gain insights on this kind of imaginative thinking by actively using appropriate materials. Students in the 9th year at a Realschule were given the following task:

**Instructions for calculating area:**
First cut out the blue pieces. Glue one piece into a notebook with graph paper, then use the squares, which are one square centimeter each, to determine the area of the piece as precisely as possible.

Here is one student’s solution:
Here we realize that the student cannot imagine the size of 240 cm², nor does she understand what the formula “length times width” has to do with calculating the area of the figure in question.

Another method for getting to the bottom of what students can (or can’t) imagine is getting them to write arithmetic stories. In a 5th year class at a Realschule, students were given the following task:

Write a problem in which the following calculations lead to the answer:

a) 12 m : 3 m =  

b) 12 m : 3 =

Out of 32 students, only 8 were able to come up with an appropriate context for both calculations. Thirteen students created inaccurate word problems in both cases or were unable to come up with any solution at all.

Examples of correct answers (in the students’ words):

➔ 12 m : 3 m  
Mr. Meyer has a board that is 12 m long. He wants to cut it into pieces that are the same length. Each piece should be 3 m long. How many pieces will he have?  
➔ 12 m : 3  
Fred is cutting a stick that is 12 m long into 3 pieces that are the same length. How long is each piece?

Examples of incorrect answers:

➔ 12 m : 3 m  
John plans to make a cycling tour with his grandfather. They want to travel 12 km. On the first day, they travel 3 km; on the second day, 4 km; and on the third day, they travel 5 km. How many meters of fence do you need if you want to surround 12 meters and leave 3 m open for a gate?  
➔ 12m : 3 =  
A slide is 12 m long. It is supposed to be taken down, though. It will be divided by 4. How often will the slide have to be divided? If my house is 12 m high, 3 parts will be repainted. How many parts still have to be painted?
When working on new material in mathematics, the ideas the students already have should be reviewed and corrected; this must then be followed by the further development of these conceptions or the establishment of new ones. This will not happen if the teacher only briefly goes into the conceptions that correspond to the new material that is being covered. The hope that students will adopt these and then be able to successfully work with them generally proves to be over-optimistic. This can clearly be seen when covering the binomial formulas, for example. The great majority of teachers introduce this topic with an illustration (see the figure on the left), but in subsequent lessons, they do not refer back to it and train students only in the formal applications of the formulas. Students frequently make errors of the type \((a + b)^2 = a^2 + b^2\); they have not internalized the idea represented by the illustration. Working with multiplication charts, as described starting on page 24, can be very beneficial in this connection.

It has been shown that if students are allowed to work on newly introduced material on their own for a while, with appropriate assignments, the result is an improvement in their imaginative thinking on the topic. This gives them the opportunity to create hypotheses, which they can then test, leading to an increase in their own knowledge. Ideally, different levels of the representation of students’ knowledge would be touched on in the assignments, making it possible for students to approach the material from different angles. Then the students present their results in class, and the teacher introduces additional aspects of the material. In this way, the concepts students have developed can be further expanded or corrected, if necessary. Only after this phase in the development of conceptual imagination, which should never be given short shrift⁴, does it make sense to introduce a more formal treatment of the material or place emphasis on routinization and automatization. Schematizing the material at an early stage will hinder the development of conceptual imagination.

To secure the imaginative concepts that have been acquired, they must be repeatedly taken up, even after students have begun to work with the material abstractly, and expanded on if necessary. „Ideally, basic concepts will develop into a dynamic, stable network of mental models, one that becomes more and more efficient through supplementation and reorganization.“⁵

---

⁴ “Wenn du wenig Zeit hast, nimm dir viel davon am Anfang” [If you have little time, take a lot of it at the beginning] (Ruth Cohn)

⁵ www.uni-regensburg.de/fakultaeten/nat_Fak_I/BIOQUA/
Examples of Implementation

Many students, even those in the upper grades, continue to have trouble determining the perimeter, area, volume, or surface area of simple figures or solids. Clearly, such students have little notion of the fact that these properties are principally determined by comparing them to a base item – that is, by measuring them. The importance of this concept is also emphasized in the educational standards, where measuring is expressly established as a core concept.\(^6\)

Calculating the perimeter or area of a figure by using a formula is an abstract business and should only be introduced once the basic idea is firmly established. Before this, students must repeatedly practice the pertinent activity on their own, for example, determining the perimeter of a figure using a ruler or the area of a figure by laying it out on a grid (or comparing it to something whose area is already known). It is not enough for individual students or the teacher to demonstrate it on the blackboard.

The appropriate approach must be used throughout all grade levels. Even in the upper grades, the use of formulas to calculate the perimeter, area, volume, or surface area of a given type of figure or solid must not take place until after students have had the opportunity to take measurements on their own.

Given a variety of basic units (see figure), students will repeatedly be required to convert one dimension into another, something that will therefore not become an end in itself.

Once students have internalized the basic concepts connected with areas, as early as the 5th grade they will already be able to roughly determine the area of irregular or unfamiliar surfaces (for example, area of a circle) by using squares measuring one square centimeter. If appropriate working methods are repeated on a regular basis during the following grades, we can expect students to have fewer difficulties solving a problem about the area of a figure, such as the one shown on page 20.

\[^6\text{www.kmk.org/schul/home1.htm}\]
Even problems like the following (from PISA 2000) would no longer represent an insurmountable obstacle:

Multiplying summed terms and factoring
Another problem area for many students is simplifying terms. It has been shown that students benefit when they are able to connect the simplification of terms to their conceptual imagination, at least to a degree. Multiplication charts, which students generally know from elementary school, are well-suited to multiplying out sums. The (area) concept of multiplication that is connected to this can (and should) be taken up and repeated and also further developed in secondary school. This topic can therefore be taught cumulatively, and the learners will recognize the consistent principle behind it.

Multiplying whole numbers:
A productive task for 5th graders might look like the following: Turn the “large” multiplication problem 386 · 48 into several “small” multiplication problems that are easy to solve.

By picturing the task as a request to calculate the area of a rectangle, students will be able to make the connection that it is based on geometry. One side of the rectangle is divided into units of 300, 80, and 6, the other into units of 40 and 8. In the end, the area of 6 pieces of the rectangle will be calculated, which when added together give the total area.
The following step toward greater abstraction shows a stage preliminary to the normal written process for multiplication:

<table>
<thead>
<tr>
<th></th>
<th>300</th>
<th>80</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>12000</td>
<td>3200</td>
<td>240</td>
</tr>
<tr>
<td>8</td>
<td>2400</td>
<td>640</td>
<td>48</td>
</tr>
<tr>
<td>14400</td>
<td>3840</td>
<td>288</td>
<td>18528</td>
</tr>
</tbody>
</table>

So: $386 \cdot 48 = 18528$

The area aspect of multiplication can be covered once again in connection with multiplying fractions or mixed numbers; students can also brush up on working with multiplication charts when they get to multiplying mixed numbers:

Multiplying fractions: $\frac{3}{5} \cdot \frac{5}{7}$

![Multiplying fractions diagram]

Multiplying mixed numbers: $3 \frac{1}{2} \cdot 1 \frac{2}{3}$

![Multiplying mixed numbers diagram]

Graphic representation of area: $3 \frac{1}{2} \cdot 1 \frac{2}{3} = \frac{35}{6} \cdot \frac{5}{6}$

So: $3 \frac{1}{2} \cdot 1 \frac{2}{3} = 5 \frac{5}{6}$
The area aspect of multiplication can profitably be applied in additional contexts.

The following assignments\(^8\) display an approach to the topic of “multiplying summed terms” that is based on this area aspect; this approach allows students to expand their ideas/imagined concepts on this topic independently.

**A painting by Richard Paul Lohse**

The Swiss painter Richard Paul Lohse created the painting “6 complementary rows of colors” in 1983.

\[^{8}\text{Adapted from Mathbu.ch 7 [Math, grade 7]. Klett and Balmer; Zug, 2002}\]

→ Describe the painting. Which regularities do you see?

→ The variables along the edge refer to the lengths that occur in the painting. With the variables, you can also express the length of pieces that have been placed together. For instance, the total length of the painting is 

\[a + b + c + c + b + a, \text{ or put differently, } 2a + 2b + 2c.\]

Explain both terms.

→ With these variables you can also express area. The area of the yellow square at the top right of the painting is described by the term \(a \cdot a = a^2\). For the green rectangle just under it the term is \(a \cdot b = ab\).
The total area of all of the orange sections is determined by the term \(2a^2 + 4bc\). You can verify it. Write out the terms for the other colors as well.

Together, the three sections of the painting excerpted below (left) form a rectangle. You could describe the area of this rectangle in two different ways: \(a \cdot (c + b + a)\) or \(ac + ab + a^2\). Both terms are equivalent. Describe the area of the detail of the painting below (right) using different terms, and also give a term for the perimeter of this section.

Look at the painting and find a rectangular section whose area is given by the term \((b + c) \cdot (b + 2c)\). Give an equivalent term for the area of this section.

Make your own rectangles from the painting. Copy them, label them with the correct variables, and describe their areas using various equivalent terms. Exchange terms with your classmates and search for the sections they belong to in the painting. Compare.

In order for the ideas the students have acquired to become firmly established, the students must work with them for an extended period of time. For this reason, they should use multiplication charts when multiplying out longer terms and factoring. Moving on to formal multiplication too soon would quickly oust the ideas they have developed.
### Multiplying polynomials:

**Example 1:** \((3 + x) \cdot (-4x + 5y)\)

\[
\begin{array}{c|cc}
-4x & 3 & x \\
5y & -12x & -4x^2 \\
& 15y & 5xy \\
\end{array}
\]

So: \((3 + x) \cdot (-4x + 5y) = -12x - 4x^2 + 15y + 5xy\)

**Example 2:** \((-3a - 5b)^2\)

\[
\begin{array}{c|cc}
-3a & -3a & -5b \\
-5b & 9a^2 & 15ab \\
& 15ab & 25b^2 \\
\end{array}
\]

So: \((-3a - 5b)^2 = 9a^2 + 30ab + 25b^2\)

### Factoring:

**Example:** \(4x^2 - 16x + 16\) \hspace{1cm} or \hspace{1cm} \(4x^2 - 8x + 16\)

\[
\begin{array}{c|cc}
-2x & 2x & -4 \\
-4 & 4x^2 & -8x \\
& -8x & 16 \\
\end{array}
\quad\quad
\begin{array}{c|cc}
-2x & -2x & 4 \\
-8x & 4x^2 & -8x \\
& -8x & 16 \\
\end{array}
\]

So: \(4x^2 - 16x + 16 = (2x - 4)^2\) \hspace{1cm} or \hspace{1cm} \((-2x + 4)^2\)

First, the students split up the terms being summed in the „result fields“. They also realize that \(4x^2\) and 16 cannot be in the same row or column, because they are not „compatible“. In a final step, the factors are identified.

### Results:

At one Realschule, 8th graders used nothing but multiplication charts to find the product of polynomials and for factoring. At the end of the year, the students did amazingly well in a test requiring them both to multiply polynomials and to engage in factoring. When the students were tested again at a later stage, they were still able to confidently use the knowledge they had gained.
Excerpt from the test:

Factor the following
a) $36x^2 - 64y^2 =
$ b) $144a^2 - 72a + 9 =
$ c) $6xy - 2x^3 - 12y + 4x^2 =

75% of the students solved problems a) and b) without any errors. What is even more astounding is that 71% of the students solved problem c) absolutely correctly using only a multiplication chart (see the figure below for students’ solutions).
Suggestions for Activity-Based Units to Train Students’ Spatial Imagination

The Importance of Spatial Imagination

Spatial imagination is generally considered an important component of human intelligence and has a considerable influence on student achievement. A variety of recent studies\(^1\) have indicated – particularly for girls – a clear connection between performance in mathematics and skills in spatial orientation. Moreover, correlations have been noted between well-developed spatial imagination and student achievement, not only in the sciences, but also in languages.

Components of Spatial Imagination

A variety of different abilities are involved in spatial imagination. H. Besuden\(^2\) distinguishes three different components in spatial imagination:

1. **Spatial orientation**
   “This is the ability to find one’s way around in physical space, either in reality or mentally.” Experience shows that left–right directionality in particular often causes people difficulty.

2. **Spatial imagination**
   “This is the ability to reproduce spatial objects, even when they are physically absent, whether through language or by drawing them.” Because imagination of this kind is linked with memory and requires conscious perception, the use of visual aids in teaching geometry, which the students should create themselves whenever possible, is of great importance.

3. **Spatial thinking**
   “This is the ability to work flexibly with the elements of spatial imagination.” Students must be aided in developing the ability to imagine a particular spatial object from different perspectives and to mentally carry out rotations or changes of position. For this, students must be able to perform and internalize activities with the objects.
The Role of Spatial Imagination in Teaching Mathematics

The importance of training students’ ability to imagine space does not seem to be sufficiently anchored in teachers’ minds, at least not as firmly as would be desirable. Math teachers seem to consider geometry less important than algebra, often relegating geometry topics to the end of the school year. Educational experts in geometry have frequently criticized the neglect of geometry within math instruction. The findings of the TIMS study also indicate that this is a weak point in the teaching of mathematics in Germany.³

An additional point is the fact that geometry instruction mostly focuses on two-dimensional figures due to prescribed curriculum standards. This has led to the conclusion “that contrary to a widely held opinion, spatial imagination plays a relatively subordinate role in supporting student achievement in geometry.”⁴

Using Learning Stages to Train Students’ Spatial Imagination

In the following, we describe learning stages that have been developed to train and enhance students’ spatial imagination. In the process, a variety of primary objectives will also be taken into consideration:

➔ Exploring, developing strategies, getting on progressively (problem-solving strategies);
➔ Collaborative learning with classmates (taking responsibility for work/working cooperatively);
➔ Presenting own approaches to problem-solving and explaining them (verbalization);
➔ Finding motivation in success, learning in a way that is playful and enjoyable.

⁴ Treumann, K., quoted in Maier, H.P.: Räumliches Vorstellungsvermögen [Spatial ability]. In “Der Mathematikunterricht” [Teaching Mathematics], Volume 45, No. 3/1999; p. 8, Friedrich Verlag, Seelze
Of the 8 stages listed below, only those printed in bold type will be presented with examples. You can find descriptions of the other stages at www.sinus-bayern.de (in German language).

Overview of the stages

An overview of all the stages
1. Orientation in a plane (patterns, geoboards, compass points)
2. Objects composed of blocks (constructing objects with blocks, oblique projection)
3. Games with the Soma cube
4. Arranging rectangular solids (side views, outlines)
5. Tilting motions
6. Knots
7. Friendship bracelets
8. Impossible figures

Stage 1: Orientation in a plane
➔ Continuing with patterns;
➔ Searching for symmetry in patterns;
➔ Copying patterns from photographs onto graph paper (Figure 1.1);
➔ Creating patterns;
➔ Creating figures on a geoboard with string or rubber bands, copying them on dot paper, and labeling them with coordinates (Figure 1.2);
➔ Coding a path on a grid by providing compass points (Figure 1.3).

Draw the decorative patterns on graph paper

Figures on a geoboard
➔ Make up your own figure on the geoboard;
➔ Draw the figure on dot paper;
➔ “Dictate” the coordinates for your figure to your neighbor (in the following example, these would be A1, B1, B2, C2, C1, D1, D2, D3, etc.).

5. From: Das Zahlenbuch 5
(Book of Numbers, grade 5), Klett and Balmer; Zug, 1999
Describing a path using compass points

Example:

<table>
<thead>
<tr>
<th>O</th>
<th>N</th>
<th>W</th>
<th>S</th>
<th>W</th>
<th>N</th>
<th>O</th>
<th>S</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>8</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 1.2

Figure 1.3

→ Explain the table and Figure 1.3.
→ Create your own figure on the geoboard, one that has the same starting and ending point, and draw it on graph paper. Once you have determined the starting/ending point, use a table to describe the figure. Dictate the “path” of the figure’s outline to your partner. He or she will try to create the figure on the board and then draw it.
→ Create tables for new figures that have the same starting and ending points, followed by creating the figure on the board/drawing it.

6. Based on: Das Zahlenbuch 6 (Book of Numbers, grade 6), Klett and Balmer; Zug, 2000
Stage 2: Objects composed of blocks
➔ Using blocks to construct objects from drawings in oblique projection;
➔ Making drawings in oblique projection of objects composed of blocks;
➔ Creating and assessing “blueprints” for objects composed of blocks (Figure 2.1, Figure 2.2);
➔ Determining the number of blocks necessary to construct an object from drawings in oblique projection (Figure 2.3);
➔ Completing partial drawings in oblique projection.

Info: Drawing blueprints for objects composed of blocks

This is an object as viewed from above. The object looks like a set of stairs and is made of 18 individual blocks. The base of the object is a square whose sides are each 3 blocks long (3 x 3 blocks). Each number indicates how many blocks are stacked on top of each other in that position.

To create a blueprint, start by drawing the base. Next, fill in the number of blocks stacked on top of each other in that position.

Construct the following objects according to the blueprints and then draw them on dot paper.

Figure 2.1

Figure 2.2
Count the number of blocks used to construct each object and draw a blueprint for each.

![Figure 2.3](image)

Complete each drawing, count the number of blocks used, and draw a blueprint.

![Figure 2.4](image)

**Stage 5: Tilting motions**

- Tilting matchboxes according to instructions provided;
- Carrying out the appropriate tilting motions mentally;
- Determining the number of spots on the face of dice based on the starting position given and the tilting motions provided in the instructions;
- Comparing rotated and tilted objects composed of blocks (Figure 5.1).

Mentally tilt a matchbox according to the instructions given in the table below and record the final position of the matchbox. Test your results by repeating the instructions with a real matchbox.

Key to abbreviations:

„U“: The top of the matchbox is facing up.

„D“: The top of the matchbox is facing down.

![Diagram](image)
Make up additional exercises for your partner (including solutions, of course).

The sixteen drawings below represent only three different objects. Which drawings show the same object? To test your answers, use blocks to build the objects yourself.

<table>
<thead>
<tr>
<th>Initial position</th>
<th>Tilting motion</th>
<th>Final position</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>F - R - B</td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>B - L - F</td>
<td></td>
</tr>
</tbody>
</table>
| D                | F - L - F - R - R | L - F - L
|                  |                | D             |

Stage 8: Impossible spatial figures

➔ Examining “impossible” spatial representations (Figure 8.1);
➔ Examining optical illusions (Figure 8.2);
➔ Drawing an “impossible” spatial figure (Figures 8.3, 8.4, 8.5).
Real or impossible? Our eyes can be fooled
Here we see “Waterfall” by M.C. Escher (1961). Where is the illusion? Which of the triangles pictured at the top do you recognize?

![Figure 8.1](image1.jpg)

What do we really see? Not everyone sees the same thing!
Old woman or young? 6 blocks or 7?

![Figure 8.2](image2.jpg)

Try to explain to your partner what you see in each of the pictures.
... and now it’s your turn!9
Follow the directions – but don’t cheat!

**Task 1:** Spend 10 seconds memorizing this unusual figure, then turn the page over and try to draw the figure from memory.

![Figure 8.3](image)

If you find task 1 too difficult, start with tasks 2 and 3.

**Task 2:** Copy the figure on graph paper according to the template.

![Figure 8.4](image)

**Task 3:** Cut out the three pieces on the worksheet and try to arrange them to form the unusual figure.

![Figure 8.5](image)

---

9 Based on: Junga, M.: *Sonderbare Figuren* [Strange Figures]. AOL-Verlag; Lichtenau-Scherzheim
How to Achieve Sustainable Learning Results

From Basic Knowledge Catalogs to Basic Concepts

The Current Situation

Students are constantly confronted with different subjects and methodological approaches. They encounter them at short intervals dictated by the class schedule. In biology the fact that only two teaching units per week are provided for this subject poses an additional problem. This makes it difficult to work on a topic coherently and thus enable intensive learning. You either have to spend a lot of time revising subject matter already discussed or you leave it at that and proceed with a new topic. The first option is more typical of chemistry instruction, the second may be more of a tendency in biology where teachers frequently introduce a new organism without
making any reference to the ones they have previously discussed. This “bit-by-bit” kind of learning can hardly be expected to bring about sustainable results. Achievement tests based largely on the material dealt with in the last lesson promote learning via the students’ short-term memory rather than processes for achieving sustainable learning results.

An interesting question in this context is how much basic knowledge students retain after a lengthy period of time has elapsed. To find out, tests covering fifth-grade basic knowledge were performed both in 1999 and 2000 at the start of the school year for sixth-grade students at the Holbein Gymnasium in Augsburg, Germany. These tests were sprung on the students without warning. The majority of the tasks involved simple reproduction of knowledge. Highest possible score in the test was 50 points. The scores notched up by the students were converted into grades in the way used in the school-leaving examination (“Abitur”) and compared with the final grades achieved for the fifth grade. The results for selected grade levels are shown in the diagrams below.

Comparison between the results of the basic knowledge test and the grades for the year

Note: Grades are based on a scale from 1 to 6 where 1 = excellent and 6 = fail.
Neither the grade point average (5e: 3.56; 5f: 4.26) nor the average score achieved (5e: 26.9; 5f: 21.4) were satisfactory. The results indicate that after their summer vacation the students were unable to recall much of the basic knowledge they had been taught in the fifth grade.

Though this is not a representative study, it invites the following conclusions: On average, there seems to be a correlation between the grade for the year and the students’ actual basic knowledge. However, the grade for the year does not allow any conclusions about the basic knowledge of individual students. A good short-term memory seems to have a much greater influence on the students’ grades for the year.

These results clearly indicate that we need to step up our efforts to achieve long-term retention of basic knowledge.

Criteria for Selecting Basic Knowledge Content

To help students acquire both basic and specialized knowledge at each grade level and retain it in the long term, the teaching staff involved first has to agree on the basic knowledge content to be provided. Alongside what is required by the curriculum, the following criteria may be helpful in selecting such subject matter:

➔ Subject matter that is important for understanding material to be taught at later grade levels;
➔ Subject matter that helps students to understand biological or chemical phenomena, processes and relationships;
➔ Subject matter of general educational value;
➔ Subject matter that contributes to basic understanding in other subjects (cross-connections);
➔ Subject matter that improves understanding of scientific issues plus methods for acquiring scientific knowledge.

The importance of these criteria is illustrated by the following example relating to teaching biology in the fifth grade.
The human skeleton

Structure:
- Skull
- Spine
- Chest (breastbone, ribs)
- Shoulder girdle (collarbones, shoulder blades)
- Pelvic girdle
- Skeleton of the arms (humerus, ulna and radius, carpal and metacarpal bones, phalanges)
- Skeleton of the legs (femurs, tibia and fibula, tarsal and metatarsal bones, phalanges)

Function: Supporting the body, protecting viscera and enabling movements to be performed (together with muscles, tendons and ligaments).

Some of the reasons for including the terms “tibia and fibula” as biological basic knowledge are:
- Basic knowledge about the structure and function of your own body is undoubtedly part of general education.
- In the 6th grade it forms the basis for comparing different classes of vertebrates and is used later to discuss the concept of homology.
- Sports teachers can draw and expand upon such basic knowledge in anatomy and physiology.

Stages of Development

First of all, the teaching staff has to agree on the basic-knowledge components for the individual grade levels. These components, plus age-appropriate descriptions, make up a basic knowledge catalog as shown in Fig. 1.

Basic knowledge index card systems may also be used. The respective term is written on the face of these cards, with a corresponding description written on the back. Students are familiar with index cards from language courses, and using such cards makes it easier for them to learn.

For teachers they are a readily available tool for testing basic knowledge.
If they only use the catalog and index cards, however, teachers run the risk of basic knowledge being viewed solely as a mere accumulation of knowledge. **Visualization of basic knowledge represents an initial opportunity to** place terms in a broader context and build understanding.

Mind maps or concept maps can be used to link up individual basic-knowledge terms introduced in the course of several lessons.

These are illustrations in which terms and conceptual relations are linked in a structured manner, thus **connecting up basic knowledge.**

---

Fig. 4: Connecting basic knowledge: Mind map, biology, 5th grade

Concept maps extend the mind maps by depicting the web of related concepts.

Fig. 5: Interlinking basic knowledge: Concept map, biology, 5th grade

“Building up basic knowledge” aims at firmly entrenching such knowledge in the minds of the students as they go through the different grade levels and helping them to maintain an overview. The basic concepts that figure both in the curriculum and in the national educational standards have also been defined for this purpose. These are fundamental biological or chemical concepts designed to help students realign the structure of what they have learned, interconnect individual aspects and develop and classify new information by themselves. This builds up an increasingly interlinked knowledge network that can be used at all grade levels (with minor age-appropriate variations). Such basic concepts can be used as guidelines running through teaching at all grade levels.

One of the basic concepts figuring in the curriculum refers to the ability of living beings to reproduce and pass on genetic information. In the 5th and 6th grade this concept necessitates a distinction between asexual reproduction and natural reproduction, with reference to vertebrates and plants. The students learn about the two-sex model and the relevance of fertilization, pollination, and mating as basic principles of natural reproduction, including the respective stages of development. These basic principles are then applied to other species, e.g. insects, in the 8th grade.

One tool for reinforcing basic knowledge is the use of the same basic graphic pattern for all grade levels.
Fig. 7: Guidelines through basic knowledge in biology: the concept of reproduction
The following is an overview of the stages of development of basic concepts from the basic knowledge catalog:

![Diagram of stages of development]

**How to Pass on and Reinforce Basic Knowledge**

Thinking of effective ways to communicate basic knowledge content to students and secure long-term retention of it is just as important as defining the actual content. Some of the ways developed by, and tested at, different SINUS schools are described in the following.

➔ **Basic knowledge notebook**: The students are given the most important terms during the lessons, they make a note of them in their basic knowledge notebook and propose a definition of their own. At the end of each teaching unit, the terms are revised and discussed, the definitions are compared and corrected as necessary.

➔ **Basic knowledge binder**: At the beginning of the school year the students receive a list of terms and definitions from previous years. New basic knowledge is supplemented from the binder and/or highlighted in color in the biology notebook.

➔ **Basic knowledge index card system**: Either the complete set of cards is given to the students right from the start, or they create such a set of cards themselves, parallel to what is taught in the classroom. The respective term is written on the face of the index card, the definition on the back.
Test-your-knowledge cards: Students pair off to test each other’s knowledge using the index cards for their questions.

Basic knowledge in electronic form: Basic knowledge of the respective subject is made available on the school website and can be accessed by the students at any time. Thus they can close knowledge gaps in any subject independently and with little effort. In addition, this variant makes it possible to use appealing graphics or even animations and to establish networks via appropriate links.

Tasks relevant for basic knowledge: Basic knowledge is extended by means of questions the students have to work on after each teaching unit or for a specified period of time.

Learning posters: Learning posters visualize the basic knowledge content of an ongoing teaching sequence in the classroom or biology lab.

Dominoes: Index cards with terms and definitions or illustrations are designed so that students can use them to play dominoes.

Games: Aside from favorites like “Who wants to be a millionaire?” or “The price is right”, many other game ideas can be used to reinforce basic knowledge. One easy example to put into practice is a basic knowledge rally. First, use index cards in different colors for different topics. The topic is written on the face, one term and its definition on the back. You also need a board with adhesive spots in different colors (corresponding to the color of the respective topic on the index card), figures, and a dice. When a player moves his or her figure to a colored field, the next player reads out the term written on the same-color index card. The student whose turn it is has to explain this term. If the explanation is correct, the figure moves three fields forward, if not, three fields back.

Puzzles: Terms relevant for basic knowledge are used for word-search puzzles, word games and crossword puzzles e.g. the “bio spielend lernen” series by Klett, Stuttgart.
Testing Basic Knowledge

Teaching experience has shown that working with basic knowledge will become even more important for students if such knowledge is not only applied in the classroom, but also included in reviews of learning objectives in a systematic and regular manner.

➔ **Activity reports**: Activity reports should describe the basic knowledge content of both the last and preceding lessons. Index cards as described above are particularly useful in the latter case.

➔ **Written achievement tests**: Previous basic knowledge content should be tested using impromptu tests, short tests and homework in a similar manner as in oral achievement tests. Grade level tests as offered by the ISB in Munich for the subject ‘Nature and technology’ in the 6th grade should also be considered in this context. Performing such tests at the end of the 10th grade could provide useful information both for students and teachers about the subject matter taught and student achievement levels over the long term in biology and chemistry.
Developing Knowledge at Different Levels of Understanding

The blue box contains four more wooden sticks than the orange one.

<table>
<thead>
<tr>
<th>x</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

\[ x = y + 4 \]

Levels of Representation after J. S. Bruner

In the 1960s Jerome Bruner developed a model of cognitive development that has largely been confirmed by findings in brain research and whose constructivist perception of learning can be regarded as being fundamental to the SINUS program. As defined by Bruner, learning is an active process. Progress in intellectual development is achieved only by actively engaging with the environment. The experience children gain in this way then has to be related to the knowledge they already have and stored. A child’s view of the world will change as he/she grows up, while dependence on external stimuli will constantly diminish. According to Bruner, the development of knowledge takes place at different levels of representation:

\[ \rightarrow \text{Enactive level: Knowledge linked to activities involving concrete objects.} \]
\[ \rightarrow \text{Iconic level: Knowledge linked to visualized ideas.} \] However, such knowledge can be retrieved without performing any specific action.
\[ \rightarrow \text{Symbolic level: Knowledge no longer linked to visualized ideas.} \]
Cumulative Learning Based on the Spiral Curriculum Concept

For mathematics, Bruner’s insights mean that introducing new subject matter should not be postponed until the final and conclusive treatment of it (usually in the symbolic mode) appears feasible, but rather should take place at earlier stages. This must, of course, be implemented in such a way that such knowledge can still be expanded at a higher level. The intellectual growth of children is a process extending over a number of years. At the same time, one level of representation will gradually be replaced by the next, albeit without being made wholly superfluous. In teaching, this repeated confrontation with a topic at different levels of increasing abstraction is also called the “spiral curriculum concept” and is included in many present-day curricula.

Example:
Introducing terms connected with variables requires caution and many years of practice at different levels to ensure that students will be able to master them with confidence at an abstract level.

These little bags contain an equal number of marbles. Three marbles are lying next to them.

Students at the age of 10 still need to manipulate, or at least connect the bags and marbles visually, in order to master the corresponding term \(2x + 3\) at a later stage. At the age of 14, the enactive mode with bags and marbles can be completely replaced by the symbolic mode. However, dealing with the subject in an active and patient manner beforehand (for example by using such bags of marbles or wooden sticks) is a must, if we want to ensure that students have a clear and unambiguous understanding of the corresponding terms. In difficult didactic situations, referring back to the enactive or iconic mode may help to achieve better understanding.

Fostering Understanding by Using Different Representation Levels in Parallel

The learning environment shown on page 51 – “Knack die Box” [Crack the Box], based on mathbu.ch 7 [Math, grade 7] – for the assignment of possible solutions addresses all Bruner’s levels of representation:

➔ **Enactive representation:** In this first stage, students use concrete material in their search for solutions to equations. How many wooden sticks are in the blue boxes and how many in the red boxes if the number of wooden sticks on both sides is to be equal?

➔ **Iconic representation:** Many students will soon be able to do without such concrete material and work solely with drawings. Red and blue rectangles represent the boxes, strokes the number of wooden sticks lying outside the boxes.

➔ **Symbolic representation:** The as-is situation of the boxes is verbally explained in such a way that it can be replicated by another student. In the next stage, variable x stands for the number of wooden sticks in the blue box and variable y for the number of them in the red box. The solution to the problem is finally shown in the table of values.

The essential thing here is that the “dummies” (bags, boxes, rectangles, letters) stand for the number of objects involved at all representation levels.

By using the different levels of representation in parallel, profound understanding of what is meant by variables and equations can be established and functional thinking prepared. The reason why students perceive learning environments like “Knack die Box” as plain and simple is the simultaneous use of different levels of representation. In the exercise section, students are encouraged to switch from one level to the other. In doing so, they find out that deriving a table of values from the box situation is easier than deriving an equation from a table of values. After some time, however, most of the students will have learned almost effortlessly how to perform the latter task as well. Functional equations are then seen as the
simplest and most clearly represented form of a function whose concrete meaning can be clearly recognized.

Using the different levels of representation in teaching means automatically taking account of the needs of the different types of learners. This way, they have a chance of finding their own approach to the relevant subject matter.

Problems that can be worked on by using different levels of representation should be an integral part of a new teaching culture enabling students to find their own individual approaches and different strategies for solutions. As students mature and continue to practice, they will automatically decide to work at the level involving the least effort. Mathematics and its formal elements are then perceived as facilitating solutions rather than an obstacle.

**Example**:

Petra hands out hazelnuts. Half of the hazelnuts she gives to Ulrike and half of the rest to Matthias, with 8 hazelnuts remaining for Petra. How many hazelnuts did Petra have at first?

<table>
<thead>
<tr>
<th>Informative figure</th>
<th>Table, trial and error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number guessed</td>
<td>20</td>
</tr>
<tr>
<td>Number for U</td>
<td>10</td>
</tr>
<tr>
<td>Number for M</td>
<td>5</td>
</tr>
<tr>
<td>Remaining</td>
<td>5</td>
</tr>
<tr>
<td>Target</td>
<td>8</td>
</tr>
</tbody>
</table>

Equation

\[ \frac{1}{2} x + \frac{1}{4} x + 8 = x \]

The suggested strategies for finding the solution operate at different levels of representation. Students aware of the fact that they can switch from one level to the next will be able to solve the problem.


Classroom Examples

The following examples show how the different levels of representation can be included in math instruction.

If formal elements of representation are left out of account, relatively complex equations can be presented and solved even in elementary school. The procedure used for this is the strategy of “working backward”, which should be used as appropriate up to and including the 8th grade. The following visual representation can be used from the 4th grade onward.

Example 1:

➔ **Textual representation** (symbolic mode):
“I am thinking of a number, multiply it by 2, add 18, divide the result by 3 and get 20. What is the number I thought of?”

➔ **Visual representation** (iconic mode):
The number sought and the intermediate results are written on the back of a sheet of paper and then used to visualize the problem on the blackboard.

➔ **Formal representation** (symbolic mode):
\[(x \cdot 2 + 18) : 3 = 20\]

Example 2:

➔ **Formal representation** (symbolic mode)
\[(44 - x) \cdot 3 = 117\]

➔ **Visual representation** (iconic mode)

To ensure fundamental comprehension when operations with fractions are introduced, much of the time available has to be devoted to the enactive and iconic modes. To begin with, for example, students are asked to translate fractions into parts of a circle or to compare, add, and subtract them using a rectangular model. Only when they find that drawn solutions are rather long-winded and laborious has the time come to switch to the symbolic level, i.e. formulate and

---

apply the corresponding rules of calculation. These can then be derived from knowledge already gained, and there is no need to use such rules as a starting point.

**Examples of corresponding exercises:**

➔ Take two fractions (represented as sectors of a circle) and use them to arrange a new fraction of a circle. Describe the latter.

➔ One half and one quarter equal three quarters. Give as many similar examples as possible!

➔ Use the parts of a circle to arrange two wholes in different ways. Then arrange 3 wholes and 4 wholes. Write down the results.

➔ I get two wholes if I ....

➔ Solve the problem of \( \frac{1}{4} \times \frac{1}{5} = \) using the rectangular model:

![Rectangular model](image)

**Procedure:** Mark one quarter with a color and mark one fifth of the rectangle with another color. What do you see? Think for a while, and you’ll be able to state the solution to the problem.

The rectangular model is also suitable for illustrating how fractions can be multiplied.

**Multiplication**

Students who have completed elementary school are already able to work on problems involving multiplication at the symbolic level, but at this point in time they have not yet gained sufficient basic knowledge about all facets of multiplication for it to be firmly
entrenched in their minds. This is one of the reasons for many
typical errors and weaknesses. Achieving a sound understanding
of how multiplication works is a major objective of mathematics
instruction up to the 9th grade. The faster this is achieved, the more
efficient teaching will be.

The following is a description of two options for dealing with
multiplication at the iconic level:

➔ The **area-related aspect** involved in multiplication:

working with multiplication tables

Multiplying two positive numbers can also be interpreted as
calculating of an area. This aspect of multiplication is incorporated
(partially in a figurative sense) when working with multiplication
charts and makes it possible to visualize the product at all grade
levels right through to multiplying terms. A detailed description
of the area-related aspect of multiplication is given on page 20 ff.

➔ The **enlargement-related aspect** involved in multiplication:

working with stair sketches

The first geometrical aspect of multiplication is usually held back
until dilation is dealt with in the 9th grade, although it would defini-
tely make sense to talk about it at a much earlier stage.

The enlargement-related aspect involved in multiplication can be
shown in a so-called “stair sketch”. This type of drawing can be
used for visual representation whenever direct proportionality
is involved, i.e. in reducing a fraction, converting when scaling up
or down, evaluating extrapolations, drawing circular diagrams,
calculating percentages, calculating the slope of straight lines, or
in connection with dilation, slope, and tangent.
The potential uses of “stair sketches” presented here is designed to encourage teachers to get their students accustomed to switching back and forth between the abstract level of a problem and a more descriptive level.

A teaching sequence focusing on the rule of three was started by assigning the students the following task:

Use your set square to find out how much higher the first floor of our school is than the ground floor.

The students rushed out of the classroom and soon came back with some useful results. A number of them had simply measured the height of one step of the stairs and multiplied this value by the number of steps.

In the subsequent discussion, this stair sketch was used to visualize the problem:

![Stair Sketch 1](image1)

Confronted later on with a problem relating to the rule of three, i.e. determining the total height of the stairs based on the height of 5 steps, the students were able to solve this problem without any problems. They knew that they first had to determine the height of one step by back calculation. Once they had understood that, it was easy for them to switch to word problems such as: “The price of 5 apples is EUR 1.50. How much do 20 apples cost?”

![Stair Sketch 2](image2)

As can be seen from the following comment by a student (8th grade), drawings of this type can be useful for students at higher grade levels as well.
Suggestions for how to use “stair sketches” for other problems:

➔ Extend $\frac{4}{5}$ to twentieths

$\frac{4}{5}$

➔ Drawing to scale

20 mm

➔ Percentages

20 %

➔ Increased base value

20 % markup

The stair sketch is an excellent tool for this type of exercise because it makes the problem more concrete than a text exercise can. Using a stair sketch, you get a much more graphic impression of the task involved.
In all these cases, the “height of a step” measured physically in the 5th grade plays an important role, as of course does the fact that all steps of “reasonable”, i.e. linear, stairs are equal in height.

Working with stair sketches means that even in grades 5 and 6 students are prepared for such topics as „linear functions“ or „dilation,“ where such drawings can be revisited to facilitate comprehension. This method points up commonalities between subject areas occurring at different grade levels that would not otherwise be apparent to the students.
How to Strengthen Students’ Individual Responsibility

Approaches to Learning by Dialogue

In the chapter entitled “Encouraging independent ways of learning” contained in the booklet “Exploring New Paths in Teaching Mathematics and Science”¹, working with learning diaries and the positive experience gained with such diaries is discussed in detail. However, numerous teachers at basic secondary schools (Hauptschulen) were skeptical as to whether this form of student activity could also be successfully employed at their respective type of school. The main reason for such doubts is the fact that students at basic secondary schools are often less proficient in terms of language skills than students of the same age at other types of school – in fact, they sometimes even have problems with reading. When confronted with mathematical problems, they may be overly challenged when asked to express their own thoughts, ideas, and insights in writing. Accordingly, they tend to reject this approach, not least because students...
who are weak in reading and writing hope for success and self-confirmation by concentrating exclusively on the calculation side of mathematics.

It turns out, however, that despite these difficulties it is possible – and indeed actively beneficial – to put greater emphasis on written verbalization of mathematical problems at basic secondary schools, too. Regular training may in fact enable students to record their train of thought in writing and, as a rule, they soon recognize the advantages for their own learning processes. Writing texts then becomes indispensable to mathematics instruction. Particularly weaker students with less self-confidence are then able to develop their own approaches to solutions much more easily. They are no longer forced to voice their possibly incorrect trains of thought out loud in the classroom, but instead can first enter them in their diary. And they receive individual feedback because their notes are regularly reviewed and commented on by the teacher. This makes learning progress transparent and has a positive effect on the student’s self-confidence.

A suitable start could be, for example, working with texts without specified tasks. The students are asked to find matching tasks by themselves, work on them together with a classmate or in groups, and afterwards present their results to the class.

Very positive experience has also been gained with the use of pictures or photographs that stimulate mathematical questions. Students first describe what they see in a picture, what they know, and what they assume. Then they formulate questions and work out solutions. All their considerations are put down in writing. The various ideas are brought together during discussion in class or, for instance, in the form of a panel of experts. Every thought, every suggestion, and every attempt to find relevant questions and ideas for solutions need to be acknowledged by the teacher. Initially, a poster with the first part of sentences that can be used to describe images provides valuable assistance for weaker students.
Another way of familiarizing basic secondary school students with verbalization in mathematics is to have them produce written summaries at the end of a lesson. This encourages the students to reflect on their own learning process and will help them appreciate their progress.

The illustration shows a student’s summary of the mathematics lesson “Introducing cubic measures”. The teacher reviews the students’ texts on a regular basis, but without correcting spelling, grammar, or errors in word order. This tolerance with regard to such errors is initially difficult, especially if students have major linguistic deficiencies.

Example:

In one ninth-grade, the students worked on the task “Reading a newspaper under difficult conditions ...”. The learning groups were only presented with the image (no title or questions). Here are two excerpts from slides that were prepared in the groups:

A newspaper has 180 cm² and a UG (unfolding) high of 180 cm (presumably the bulk of paper, or the “footprint” of one newspaper). The boy who sits on the newspaper is about 30 cm tall. We have to use this key twice to get the approximate height of the newspapers:

\[ 2 \times 90 \text{ cm} \times 180 \text{ cm} = 36 \times 180 \text{ cm} \]

The width of a newspaper is approximately 1 meter, so the boy sits at a height of approximately 3.8 meters.

Volume of all newspapers:

\[ V = \text{a} \times \text{b} \times \text{c} \]

\[ V = 3.1 \text{ cm} \times 1.8 \text{ cm} \times 180 \text{ cm} \]

\[ V = 123.58 \text{ cm}^3 \]

- One newspaper weighs 180 grams.

\( 120 \text{ newspapers} \times 180 \text{ grams} = 21,600 \text{ grams} \)

The stack weighs 21.6 kilograms.

- If the cyclist had to travel with (= deliver) the newspapers, then he has to transport the newspapers and the boy.

\( 21.6 \text{ kg} + 30 \text{ kg} = 51.6 \text{ kg} \)

He must transport 51.6 kg on the last wheel.

\[ \Rightarrow \text{It cannot work because it is too heavy} \]

- because it is too high and it will begin to sway.

This is perhaps made clear by the following extract in which a student with dyslexia describes his approach to calculating the volume of a triangular column:

“Today I put cubes together. The cubes had side lengths of 1 decimeter, 1 centimeter, and 1 millimeter.

I was fascinated to find that you need 1,000 dm$^3$ for 1 m$^3$. Now I know that 1,000 dm$^3$ fit into 1 m$^3$. I found it fascinating that the distance here is 1,000 because it is too for a square. It was real very interesting.”

“Correction” of the students’ work primarily aims at recognizing valuable trains of thought and assessing the personal learning success of each individual student. Linguistic errors are therefore of secondary concern. The student’s path to the result and the reasons for mistakes made on the way become the most interesting aspect of daily teaching activity. In a remark at the end of the respective entry, teachers comment on what the students have written, thus demonstrating to each of them their interest in their thinking and learning process. In this manner they accompany and support the students along their individual learning path.

Teachers often claim that instruction of this type is hardly practicable due to lack of time. However, the concentration and perseverance that weaker students in particular invest in
writing down their thoughts on paper, once they have overcome their initial reluctance, dispel all doubts. When students deal with something intensively, they will not forget it again so quickly.

The Math Diary: Reflecting on Personal Progress in Mathematics

The booklet “Exploring New Paths in Teaching Mathematics and Science”\(^1\) goes into great detail on the connection between learning diaries and the promotion of personal responsibility for learning. Accordingly, only the most important advantages of this form of student activity are briefly summarized here:

➔ Fostering learning in an independent and self-managed manner;
➔ Providing assistance in dealing individually with the material to be learned and in designing individual learning processes;
➔ Promoting the students’ ability to express themselves in language and to argue cogently;
➔ Establishing a dialogue between each student and the teacher (learning by dialogue).

A more sophisticated approach to the use of a learning diary in mathematics instruction ("math diary" for short) is described in the following. In addition to the aspects described above, the focus here extends to

➔ regular reflection on material already dealt with and the student’s own learning processes;
➔ learning from mistakes.

\(^1\) Bavarian State Ministry of Education and Cultural Affairs: Weiterentwicklung des mathematisch-naturwissenschaftlichen Unterrichts – Erfahrungsbericht zum BLK-Programm SINUS in Bayern, Munich 2002 [Enhancing Teaching in Mathematics and Natural Science]
How to Use the Math Diary

Three different types of entries are provided for in the math diary: weekly reviews, analysis of tests, and performance of assignments.

Each week students are asked to prepare an entry at home in their own words. In doing so, they reconsider what has been learned, what proved to be difficult, where the new material can be applied, what subject matter that has already been introduced is not so easily recalled, etc. In short, new subject matter and individual learning processes are reflected on in a targeted manner. The students are given the opportunity to demonstrate particular commitment, e.g. by collecting additional information from the Internet or from newspapers and by working out their own examples and new tasks.

Weekly reviews

This week we are calculating areas within a coordinate system with the help of a determinant equation. It was partly repetition of what was learned at the beginning of the ninth grade. In some cases we also did calculations with an unknown x or y value of a coordinate point.

But most of the exercises were somewhat more difficult. The marking corners moved along a straight line that was specified by a linear equation. So we did not have the x and y value of a coordinate point, but the y value was indicated in the form of a linear equation.

The special thing about the new exercises was that it was hard to find determinantal equations with two unknowns. In this case I selected an exercise from the book.
**Analyzing homework and tests**

Within one week after the discussion of mathematics homework and grade level tests in class (with sample solutions), the students are instructed to analyze their own work at home in the math diary. They are required to go through the individual tasks and consider where and what type of errors they made. As a result, they should find out where weaknesses remain so they can be eliminated either independently or with support from the teacher.

![Image of a math problem and solution]

**Performing assignments**

Like every other learning diary, the math diary is also used for the independent performance of assignments in mathematics classes. Here it is not so important how far the students get. It is the written explanations of how the problem was solved that play an important role. Students unable to make progress on their own can ask for advice and assistance from classmates at a meeting table, but without their working materials. In the diary they note down who they approached for help.

The students are personally responsible for their diaries. They should be neat and orderly and there is no reason why the students should not be imaginative and creative into the bargain. By no
means should it be a diary containing only copied material, and
definitely not a joint effort involving friends, tutors, mothers,
fathers, siblings, or other relatives.
Parents are asked to support their children in working indepen-
dently. They are expected to show interest in the work done by their
children in the learning diary, but they should not intentionally or
inadvertently increase their children’s dependence through constant
reminders or their own participation. It is part of a learning process
for children to take the consequences themselves if they have failed
to take care of something in a regular, proper, and timely fashion or
if they simply couldn’t be bothered.

Providing Feedback
to the Students

To provide feedback to the students, the teacher inspects and eva-
ulates the entries in the math diary, but does not correct them in
the sense of identifying and rectifying finding mistakes. Following
Ruf/Gallin², the evaluation is carried out using checkmarks. For this
purpose, the criteria below (which are familiar to the students) are
applied:

**0 checkmarks:** You have not performed the assignments with
sufficient care (or not at all).

**1 checkmark:** You have worked on the material and/or tasks in a
careful and orderly fashion.

**2 checkmarks:** You often have interesting thoughts and ideas, you
take trouble with your explanations, you arrange things clearly, you
work pretty confidently on your assignments.

**3 checkmarks:** You show particular commitment, you have unex-
pected insights and/or make intelligent mistakes and have a remark-
able perception of problems; you work on tasks very confidently,
your presentation is particularly successful.

The time required to do this can be kept within reasonable limits
by examining several entries together.

² Ruf U., Gallin P.: Dialogisches Lernen in Sprache
und Mathematik. 2 volumes,
Kallmeyer; Seelze 1998
[Learning by dialogue –
languages and mathematics]
Assigning Grades to the Math Diary

Aside from conveying learning content, the math diary promotes skills that are even more important, such as independent and consistent work (also with the textbook), verbalization, text comprehension, problem-solving skills, and the assumption of responsibility for progress (including closing gaps). Conventionally, these skills are hardly ever taken into account when giving grades. Also, grades are normally based only on results achieved at the end of a learning process. Using the learning diary, however, makes it possible to trace and evaluate processes over a longer period. For this reason the students’ achievements with the math diary should be an integral component of the mathematics grade. How this is actually implemented is up to the teacher. Of course it must be transparent, in line with the respective school regulations.

For a procedure compatible with the valid school regulations for secondary schools, go to www.sinus-bayern.de.

Experience Gained

➔ Initially, there was major uncertainty among the students. Many of them wanted clear guidance and would have preferred to have a sample diary that they could copy. However, the students were deliberately not provided with such assistance. They were supposed to find their own approach and in fact did so in the course of time.

➔ An essential motivation for many students involved “being allowed to do it the way they wanted to”. They were given an opportunity to mobilize their individual strengths, such as creativity, imagination, pleasure in designing, and clear organization of work, all of them factors that otherwise receive too little attention. Many students were capable of astounding achievements and rightfully proud of their work.

➔ At the beginning, some students had serious difficulties when it came to writing down their thoughts in a comprehensible manner. However, most of them improved their verbal skills over time.

Even students who did not particularly like mathematics were able to develop a basic understanding of math problems by dealing with them in their diary. Some of them were able to improve as a result. A verbal survey in May showed that about half of the students felt they had personally benefited from keeping the diary. They were
more confident about expressing themselves and better able to retain what they had learned.

➔ There was considerable approval on the part of the parents as long as the grade for the math diary had little or no effect on the overall mathematics grade (for example, as an oral grade). Giving it greater weight led to skeptical reactions in the initial phase. Parents feared that children would be helped outside school or that some might copy from each other, so the grades would not reflect the individual achievements. These doubts were dispelled at the parent-teacher conference and by a round-robin letter reassuring the parents that the children would be keeping the diary in their own words. If adults tell them what to say, this is usually easy to identify. In retrospect, we recommend that the diary project and the associated learning and educational goals be described precisely in a letter at the beginning of the school year.

➔ Reviewing math diaries is time-consuming unless teachers are used to it. As they gain experience and develop a trained eye, the amount of time required decreases.

➔ It is advantageous if another colleague has students doing the same tasks in a parallel class. Then the teachers can consult with each another.

➔ By looking at math diaries, teachers become better acquainted with the personality of the individual students and their specific skills. Reading the diaries can in fact be enjoyable.
Establishing a Range of Methods

From Teacher Predominance to Methodological Diversity

Teaching Methods in Biology and Chemistry Classes

In 2003 an educational journal\(^1\) described the following procedure as a frequent example of the way science subjects get taught. Readers can decide for themselves whether the somewhat exaggerated account is accurate.

---

\(^1\)Jahresheft XXI „Aufgaben“. Friedrich Verlag; Seelze 2003, p. 116–118 [Yearbook, “Tasks”]
<table>
<thead>
<tr>
<th>Classroom teaching phase</th>
<th>Description of classroom teaching phase</th>
<th>Student/teacher activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Checking homework</td>
<td>The teacher reads out the correct solution for the homework and devotes his/her attention more or less to the students.</td>
<td>reading aloud, listening, reproducing</td>
</tr>
<tr>
<td>Progress review</td>
<td>Question-answer game by the teacher</td>
<td></td>
</tr>
<tr>
<td>Introduction/Motivation</td>
<td>The teacher says what they will be dealing with today, asking: “How far did we get last time?”</td>
<td>asking (appropriate) questions, copying</td>
</tr>
<tr>
<td></td>
<td>Often prompted by the teacher, the students identify a key issue, which is written on the blackboard and copied into their notebooks.</td>
<td></td>
</tr>
<tr>
<td>Study and development phase</td>
<td>Classroom discussion develops on the basis of questions under the teacher’s guidance.</td>
<td>following carefully, listening, answering, reading</td>
</tr>
<tr>
<td>Defining the problem</td>
<td>Between them, the teacher and the students summarize the results on the blackboard and then in the students’ notebooks; this often continues until the bell rings; after that homework is assigned.</td>
<td>reading aloud, listening, discussing, copying, summarizing</td>
</tr>
</tbody>
</table>

What we have here, from a didactic point of view, is the predominance of instruction. Knowledge is systematically organized and taught by experts. The students absorb knowledge; their activities involve following closely and understanding the subject matter. This form of teaching is suitable primarily for very difficult or complex subject matter where pronounced guidance on the part of the teacher is required.

Nowadays, however, there is increasing focus on acquisition of competence by the students. The goal is “scientific literacy”, i.e. the “ability to apply scientific knowledge, identify scientific issues, and draw conclusions from evidence in order to understand and make decisions concerning the natural world and the changes imposed on it by human activity.” Accordingly, students have to be given a regular opportunity to develop strategies and solutions themselves. “Experts agree almost unanimously that in the everyday school context inadequate emphasis is placed on the kind of student activity (both in and outside class) that calls for greater responsibility on the part of the students and demands a higher degree of self-organization – particularly in mathematics and science classes...

As students get older, they should be expected to regulate their learning themselves.\(^3\)

**Constructivist learning environments** offer useful ways of achieving this. Students solve problems largely on their own and acquire knowledge in the process. The teacher acts as a consultant to be turned to if they need help. Student activities compatible with this approach include projects or workshops. They provide scope for individual ways of dealing with a problem and enable students to determine their own work pace. They usually also require more classroom time.

### A Different Way of Teaching: A Synthesis of Construction and Instruction

Learning environments based on the sandwich principle (Wahl, 2005\(^4\)) enable teachers to take advantage of the strengths of both approaches. This involves regularly interspersing phases of knowledge transfer with phases of individual learning. Class teaching that uses this method to break away from the strict instruction-only approach can be illustrated as follows\(^5\):

<table>
<thead>
<tr>
<th>Information</th>
<th>Exercise 1</th>
<th>Development 1</th>
<th>Exercise 2</th>
<th>Development 2</th>
<th>Exercise 3</th>
<th>Repetition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lecture-style teaching</td>
<td>Individual work</td>
<td>Group instruction</td>
<td>Student experiment</td>
<td>Lecture-style teaching</td>
<td>Group work</td>
<td>Structuring Lecture-style teaching</td>
</tr>
<tr>
<td>Texts</td>
<td>Work with partner</td>
<td>Lecture-style teaching</td>
<td>Group work</td>
<td>Group work</td>
<td>Mini-projects</td>
<td></td>
</tr>
<tr>
<td>Video</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Materials</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Teacher lecture</th>
<th>Demonstration</th>
<th>Experiments</th>
<th>Complex assignment</th>
<th>Teacher experiment</th>
<th>Complex assignment</th>
<th>Concept map</th>
</tr>
</thead>
<tbody>
<tr>
<td>Texts</td>
<td>Drawing assignment</td>
<td>Teacher-student discussion</td>
<td>Applications</td>
<td>Student experiment</td>
<td>Applications</td>
<td>Mind map</td>
</tr>
<tr>
<td>Video</td>
<td>Thinking assignment</td>
<td></td>
<td>Context assignment</td>
<td></td>
<td></td>
<td>Summary</td>
</tr>
<tr>
<td>Materials</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Reiteration</td>
</tr>
</tbody>
</table>

The constellations and teaching methods indicated in the above scheme should be understood as a range of potential options.

The key point is that the students receive structuring aids through phases of instruction between which they are given broad scope to grapple with the subject matter at their own individual learning pace, taking into account different pre-acquired skills. This teaching method is suitable for acquiring learning strategies, competencies, and systematic specialized skills. It can easily be adapted to the
achievement level of the class and age bracket: students with fewer pre-acquired skills prefer “thin sandwich layers”, i.e. shorter collective learning phases followed by shorter processing phases, while “thick sandwich layers” are more suitable for students with more extensive pre-acquired skills.

In the following, the sandwich principle is illustrated with reference to two examples.

**The human body – a complex networked system**

**Preliminary didactic considerations**

**Time required:** 3–4 class periods

**Incorporating the topic into the curriculum**

The teaching unit is taken from the subject area “NT 5.2.2 The human body and keeping it healthy” in the Bavarian curriculum for “Gymnasium” (grammar school) and is relevant for the sections “Principle of blood circulation” and “Interconnection of bodily activity – nutritional requirements – breathing frequency – heartbeat frequency”.

**Knowledge acquired from previous classes**

Organs and organ systems, such as heart, lungs, skin, blood, digestive organs, and kidneys with their functions

**Objectives of the teaching unit**

As set out in the aims of the curriculum, the students examine the human body in detail and find out about the interactions between the organs. In this process they recall knowledge gleaned previously and combine this knowledge in a new way that encourages cumulative, comprehension-based learning. A concept map is extremely useful in this context. It requires and encourages initiative on the part of the students and thus facilitates a constructivist approach. Use of “help cards” also makes for achievement-based differentiation.

On the basis of a topic related to everyday life (meaningful context), students can work out the existing interrelationships by themselves. In doing this, they use their pre-acquired skills and experience their own (increasing) competence. In addition, the teacher can encourage health-conscious behavior without any trace of a “holier than thou” attitude. By using a poem as a source of information and

---

having the students carry out mathematical calculations, the teacher also gives them an interdisciplinary context for their work.

Classroom procedure for the teaching unit
Instruction
As an introduction, “Fred Grimes” is presented, a fat man who smokes, devours great quantities of greasy food, never exercises, and goes everywhere by car. The teacher reads the corresponding poem aloud and then passes it out to the students.

Two billion and thirteen times beat the heart of Fred Grimes
Two billion and fourteen now but Fred doesn’t care anyhow, sits in the office, no sports for him, only goes by car, never to the gym. He smokes and eats greasy food forgets his heart in a good mood. This organ is now working hard pumping blood with all its might but after he’s eaten all that lard his heart will soon lose the fight. For beat sixteen it still didn’t slip but number seventeen made it tip. Fred Grimes, dead now on his back, is another victim of heart attack.

The text tells the students that the heart stops beating after a heart attack. During the discussion the question arises as to why failure of an organ in the human body poses a problem in the first place. The students are familiar with networked systems from their everyday lives (heating, plumbing, transport systems). So they can easily appreciate that the failure of one organ is bound to affect all the other organs as well since they are linked by the blood circulation system, etc.

Construction
After the topic for the classroom period, “The human body – a complex networked system”, has been outlined, the students use a concept map to analyze the interrelationship(s) between the

individual organs. The map later is given to them as a pre-drawn diagram, the terms to be inserted are entered in the form of two text fields (boxes for nouns, arrows for verbs). The extracts shown convey an impression of the complete documents (available at www.sinus-bayern.de in German language). Since drawing up the concept map is very demanding and requires a high degree of concentration, perseverance, ability to think in abstract notions and networked concepts, students can work together with their classmates to foster cooperation and mutual support. To encourage independent exploration of the subject matter, on the one hand, and provide students who need further assistance with the help they need on the other, a useful methodological ploy is to combine the concept map with “help cards.”

Five help cards, each containing a brief text with information on the pre-acquired skills to be activated are laid out ready for use on the teacher’s desk. The students can take a card as needed and continue to work with the information provided. If one card is not sufficient to find the solution, the other cards can be used (the numbering of the cards only serves to distinguish them, they can be taken in any order).

Fig. Example of a concept map
A work group is asked to write its solution on a slide and explain the results orally. The predetermined structure makes it easier for the students to speak about the subject without preparation.

**Instruction**

To take up the initial topic of the teaching unit again, the teacher asks what led to Fred Grimes’ heart attack. The students have to use their knowledge and skills from their English course to identify the relevant points in the “literary” text (poem) relating to the development of a heart attack. In the retention phase the results are written on the blackboard.

To check the learning outcome at the end of the teaching unit, the teacher shows a picture of an emergency situation with the question “What can you do when someone has a heart attack?”

The students are asked to use what they have learned to make suggestions on how to deal with the consequences of cardiac arrest. The heart stops pumping, so the blood has to be pumped artificially through the body by means of cardiac massage. Since the unconscious person has also stopped breathing, oxygen has to be supplied to the blood through artificial respiration.
Finally the students have to deal with the question of Fred Grimes’ age. Working individually, they tackle this assignment by activating skills from their math classes. At the same time, they practice dealing with numbers and arithmetic operations.

60 x 70 beats = 4,200 beats an hour
4,200 x 24 = 100,800 beats a day
100,800 x 365 = 36,792,000 beats in 1 year
(rounded to 37,000,000 beats)
Mr. Grimes’ age:
2,000,000,000 : 37,000,000 = approx. 54 years old

Chlorine in the swimming pool?

Preliminary didactic considerations
Time required: 2 class periods

Incorporating the topic into the curriculum
The teaching unit was developed for the Bavarian „Gymnasium“ (grammar school; senior high) course and makes it possible to link chemical, ecological, and economic questions in a way recommended in the descriptions of the objectives for the subject areas “Electrochemistry” and “Chemical Plant”. There are references to the following subject matter:

- Electrolysis as a forced redox reaction (measurement of decomposition voltage and overvoltage)
- Quantitative treatment (Faraday’s laws)

Example from upper secondary school level
(US equivalent: senior high school; UK equivalent: 6th form)
Applications in technology; electrolysis of alkali metal chlorides (methods, economic significance, pollution, methods of reducing emissions, recycling)

In an eight-year “Gymnasium” course, the teaching unit is applicable for the curriculum section C12.3 “Redox equilibrium, electrolysis”.

Pre-acquired knowledge
To ensure that the students can work with the texts meaningfully, the curriculum subject matter pertaining to redox equilibrium, standard potential, and electrolysis as a forced redox reaction should already have been discussed.

Objectives of the teaching unit
The systematic alternation between instruction and construction in this sequence of lessons requires the students to take an active and variegated part in classroom activity and immerse themselves intensively in the relevant subject matter (conference of experts, creating projector slides and posters, letters to the editor, written statements of opinion).
At the same time, it is an opportunity to acquire the subject-related competencies called for in the national educational standards for chemistry. Not only are specialized skills necessary here, but also an in-depth ecological and economic assessment based on the chemistry of the problem.

Classroom procedure for the teaching unit
Construction
After reading the following article and subsequently exchanging ideas in groups to clarify the subject matter, the students jointly formulate the question:
Does it make chemical and ecological sense to use a substitute for chlorine in swimming pools?

Material 1

**Bathing like Cleopatra**

The new Stiftland Reha therapy pool dispenses with chlorine

Mitterteich (xtk). The best for his guests – that’s what Wolfgang Haas wants. Now the owner of the Mitterteich “Stiftland Reha” (rehabilitation center) has once again invested money to spoil his customers, guests, and patients with a brine pool.

The avid swimmers in the medical bathing department of the rehabilitation center can do their laps at a temperature of 32 degrees centigrade. The new feature: Haas dispenses with chlorine additives in the pool water, using pure natural salt from Bad Reichenhall or the Dead Sea instead. Through electrolysis the salt turns into hypochlorous acid from which salt forms after the disinfection process. The pure oxygen and hydrogen produced in this process improve the water quality...

**Instruction**

The teacher demonstrates and explains the common-salt electrolysis process. At the same time he/she makes reference to the formation of hypochlorite during the mixing of the reaction products.

**Construction**

The students work with materials 2 to 4 (presented below) by simulating a conference of experts. The results of the individual groups can be summarized on projector slides or posters.

**Brief description of procedure observed by a conference of experts**

1st step:

→ Groups of experts (in this case groups A, B, and C) consisting of 3–4 students each are formed.
→ Assignments are carried out.
2nd step:
After the groups of experts have carried out the assignments, mixed groups are formed, each containing one representative of the different groups of experts. The students report to one another on the work done in their respective groups and can now tackle a joint task (e.g. answering the question “Does it make chemical and ecological sense to use a substitute for chlorine in swimming pools?”).
**Hypochlorous Acid**

It forms when chlorine is discharged in water. However, the back reaction forming chlorine water is more advantageous in energy terms. That is precisely why you only find bleaching and cleaning agents with “chlorine-free” on the label in shops nowadays. It was a response by the industry to household accidents. Hypochlorous acid has a pK₅ value of 7.5 and is available only as an aqueous solution. The solution has a green-yellowish color and smells of chlorinated lime. It makes little sense to produce hypochlorous acid for stockpiling purposes because it decomposes (slowly in the dark and rapidly in sunlight) into hydrochloric acid and oxygen. The atomic oxygen resulting initially from decomposition is extremely reactive and therefore has a disinfecting and bleaching effect. The salts of hypochlorous acid have this effect as well.

**Various cleaning agents are available for cleaning fountains. They can be classified into the following categories:**

1. Cleaning agents containing chlorine, i.e. cleaning agents that release chlorine and hypochlorite
   1.1 Javel water (also known as Eau de Javel, Javelle water) is the most frequently used cleaning agent containing chlorine. Javel water consists of an aqueous solution of potassium hypochlorite, …
   1.2 Calcium hypochlorite Ca(ClO)₂ is also used in place of Eau of Javel and in the same concentration reacts more intensely than Javel water, …

The risks involved in the use of fountain cleaning agents can be summarized as follows: compounds that release chlorine, hypochlorite, and oxygen are strong oxidation agents and act as a bleaching agent in a concentrated solution…

**Chlorine Gas Alarm at Indoor Swimming Pool**

Chlorine is supposed to protect swimming pool guests against germs. That’s why most swimming pools disinfect their water with chlorine gas. But if the aggressive substance is inadvertently released into the air in the event of an accident, it can lead to severe harm for pool users. So to protect people against accidents with chlorine gas, safety measures have to be further improved at many locations.
Shortly before 12 noon on 17 May 1999, an emergency alarm goes out to the Grosshöchstetten BE district hospital. A serious chlorine gas accident has occurred at the local indoor swimming pool. Doctor Heinz Burger and his team only have a few minutes to organize medical and psychological care for the injured. Most of the first accident victims are taken to the hospital by private helpers. The people affected are primarily schoolchildren and older swimming pool guests who were not able to get to safety from the acrid-smelling gas in time. Because the corrosive toxin has penetrated deep into their lungs, the patients suffer from acute shortness of breath and a convulsive cough that also triggers nausea in some cases. Some of the elderly victims react with severe asthma attacks and in their fear put up resistance against the oxygen masks offered. ...

Conflicting Objectives Between Hygiene and Chemical Safety

Wouldn’t it be easier to consistently dispense with dangerous chlorine gas in critical places frequented by large numbers of people – such as public swimming pools? For hygienic reasons Urs Müller sees no real alternative to chlorination of pool water. Urine, traces of excrement, secretions from wounds, perspiration, and other bodily excretions in pools create a dangerous breeding ground for the spread of bacteria. In the warm and damp environment of a swimming pool, furthermore, the growth conditions for pathogens are ideal. ...

Chlorine – a Curse and a Blessing

Chlorine is a very reactive chemical substance that only occurs naturally in connection with other elements. Because of its powerful toxic effect, concentrations of three grams of pure chlorine per cubic meter of air lead to death after only a few breaths. ...

Instruction

The results of the groups of experts are dealt with more intensively by means of demonstration experiments and their evaluation.11

➔ Presentation of an alkali hypochlorite solution through electrolysis
➔ Presentation of an alkali hypochlorite solution composed of chlorine water and lye
➔ Verification of bleaching and oxidizing effect of the hypochlorite solutions

Construction

The students are now able to make a summary assessment of the newspaper article from chemical, medical, and ecological view-
points. Ways of doing this include drafting a letter to the editor or writing an official statement of position.

These examples serve only to provide ideas. Other good suggestions can be found in the following publications:

➔ Unterricht Chemie, Aufgaben, Heft 82/83, August 2004, Friedrich-Verlag Berlin [Teaching Chemistry]
➔ Unterricht Chemie, Naturwissenschaftliches Arbeiten, Heft 76/77, August 2003, Friedrich-Verlag Berlin [Teaching Chemistry, Scientific Work]
➔ Unterricht Chemie, Methodenwerkzeuge, Heft 64/65, September 2001, Friedrich-Verlag Berlin [Teaching Chemistry, Methodological Tools]
➔ Unterricht Biologie, Aufgaben: Lernen organisieren, Heft 287, September 200, Friedrich-Verlag Berlin [Teaching Biology, Exercises: Organizing Learning]
➔ Offene Lernformen im Chemieunterricht, Akademiebericht Nr. 395, Akademie für Lehrerbildung und Personalführung Dillingen 2004 [Open Learning Forms in Chemistry Class]
➔ Methoden-Handbuch Deutschsprachiger Fachunterricht (DFÜ), Varus Verlag Bonn 2003 [Methodological Manual for German-Speaking Classroom Instruction]
➔ Unterricht Chemie, Kompetenzen entwickeln, Heft 94/95, April, Mai 2006 [Teaching Chemistry, Developing Competencies]
➔ MNU 59, 2006, Heft 5
The starting point for further development of classroom instruction in the framework of the SINUS Transfer program was the identification of problem areas noted by the teachers involved in the course of an analysis of student achievement and reflection on their own teaching. Subsequently, strategies for eliminating these shortcomings were elaborated and specific classroom materials developed for the purpose. Testing these materials and exchanging views on them afterwards turned out to be very productive.

Many teachers criticized the passive attitude of the students at middle and upper secondary school levels, coupled with an inhibition about making mistakes. In addition, they complained that many students cannot solve old basic problems, especially when they are incorporated into a more complex context, and subject matter is dealt with at too abstract a level too early although the students have not yet been able to develop any specific conceptions about it.

The goal the teachers set themselves was to effect a positive change in these problem areas by increasing independent activity on the part of students. Independent activity is viewed as the foundation for acquiring new skills. Students are to retain what they have learned on a long-term basis by intensively preoccupying themselves with prepared material.

As part of a project entitled “Tasks encouraging student activity”, numerous materials were developed as a contribution to achieving the goals referred to above. Starting points included working with models, finding one’s own approaches to the solution, mathematical games, use of interactive worksheets on the computer, home experiments, and material-aided exercise phases. Some of these materials are presented in the following. Additional materials (in German language) are available on the Internet at www.sinus-bayern.de.

Games and Contests

Especially at lower secondary school level, it is easy to motivate students to work if the engagement with mathematical subject matter is organized as a game or contest.
**Example 1:**

A mathematical contest based on a TV quiz that was very popular in the past ("Der große Preis" – “The Big Prize”) was designed specifically as a way of starting off the school year at 5th grade level. It involves using an exercise sheet so the students can apply and repeat their pre-acquired skills from elementary school (primary school) regarding addition, subtraction, multiplication and division of natural numbers, geometry, calculations with quantities, and text problems.

<table>
<thead>
<tr>
<th>Addition/ Subtraction</th>
<th>Multiplication/ Division</th>
<th>Geometry</th>
<th>Calculating with quantities</th>
<th>Text problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Enter the missing numbers</td>
<td>Calculate: 5000 - 700</td>
<td>The numbers are mirrored here. What are the numbers?</td>
<td>John, Steve, Tim, and Mary are running a relay race. John runs 30 s, the others 32 s, 35 s, and 38 s. How many minutes and seconds does it take the four altogether?</td>
</tr>
<tr>
<td>20</td>
<td>What is the number called?</td>
<td>Divide 833 : 17</td>
<td>A cuboid has ... corners, ... edges and ... faces</td>
<td>Mr. Hall has ordered 100 bales of straw and 85 bales of hay. One bale weighs 18 kg. How much do all bales together weigh?</td>
</tr>
<tr>
<td>30</td>
<td>What numbers are missing in this series?</td>
<td>45 - 1359 =</td>
<td>What geometric shape does the sectional area have?</td>
<td>Ms. Miller needs an eighth of a liter of milk for a cake. How many milliliters of milk does she need to bake 2 cakes?</td>
</tr>
<tr>
<td>40</td>
<td>765432 – 332667 =</td>
<td>3 - 604 – 4 =</td>
<td>Draw a network for this cuboid and color the opposite faces the same!</td>
<td>Harry gets pocket money of 1.40 euros a week. How much pocket money does he get in 10 days?</td>
</tr>
<tr>
<td>20</td>
<td>Enter the missing number:</td>
<td>690 + .......... = 762</td>
<td>You have to calculate 98 - 7 in your head. How do you go about it?</td>
<td>Jenny buys three cartons of milk for 75 cents, a pack of butter for 1.27 euros, and 6 lollipops for 15 cents. How much does she get back if she pays with a 10-euro bill?</td>
</tr>
<tr>
<td>60</td>
<td>Enter the missing numbers:</td>
<td>53738 : 7 =</td>
<td>Transfer the figure to a squared sheet and draw the corresponding mirror image!</td>
<td>Ms. Bird orders a carpet from a catalog for 564 euros and a TV set for 1899 euros. She pays the bills in three installments of 109 euros each. She wants to pay off the rest in the next 6 months with monthly installments. How much is each installment?</td>
</tr>
<tr>
<td>70</td>
<td>Enter the missing numbers:</td>
<td>Estimate: 6000 · 58</td>
<td>What does the letter look like after a quarter turn?</td>
<td>Ms. Bird orders a carpet from a catalog for 564 euros and a TV set for 1899 euros. She pays the bills in three installments of 109 euros each. She wants to pay off the rest in the next 6 months with monthly installments. How much is each installment?</td>
</tr>
<tr>
<td>80</td>
<td>What number has to be added to 7918 to get 10003?</td>
<td>If I divide half of a number by 15 and add 460 to this result, I get 1060. What is the number?</td>
<td>How many cubes were used to make this cube structure?</td>
<td>An exhibition was visited by a total of 952 persons on 4 days. A fourth of the visitors came on the first day, a sixth on the second day. On the third day half as many persons came as on the first and second day together. How many persons came on the fourth day?</td>
</tr>
</tbody>
</table>
They work in groups and try to win as many points as possible via “calculation” within the specified time. They can choose problems of different degrees of difficulty from the 5 categories. The more difficult a problem is, the more points the students can get by solving it (see 1st column in the table of problems). One option would be to stipulate that at least two problems in each column have to be worked on. The solutions are entered in a worksheet containing the same table as the table used for the game. The teacher then collects and evaluates this sheet.

Game time is about 30 minutes, another 1-2 classroom periods are necessary for a detailed follow-up discussion in which the teacher should go into detail on any weaknesses observed.

Developing Work Materials Independently

The students experience and achieve a high level of independent activity when they make work materials entirely or partly on their own. A suitable option in physics class is for students to build simple devices independently, thus boosting their understanding of how these devices work and hence their comprehension of fundamental principles operative in physics. The following example is an appropriate exercise.

In mathematics, too, there are various ways of applying this procedure. For instance, it has proved advantageous to get the students to make basic geometric bodies that can then be used for illustration purposes. Very positive outcomes were also achieved by putting together and using a “geoboard,” as described below.

**Example: Dynamometer**

Home experiment: Planning, building and testing a simple dynamometer developed by Alfred Schmitt. The contest held among students of the introductory physics class was advertised as follows:

“Plan, build, and test a simple dynamometer at home. Also define the area of application for your device.”
Apart from the possible grade and the chance of winning prizes, the young “researchers” had the incentive of demonstrating both their creativity and the precision and functionality of their devices to the jury, consisting of the school’s advanced physics class and student teachers from the university physics department. The students grappled with the concept of “force” in their homework and not only determined forces theoretically, but also experienced forces of varying intensity while checking their dynamometers. At the same time, the students were introduced to scientific working methods, improving their devices via repeated measurement and recording their procedures.

Example: Geoboard

Before the students examine figures, areas, and angles in geometry class, they put together at home a so-called geoboard, which they are frequently already familiar with from elementary school. They use materials from a do-it-yourself store and are provided with instructions.

In class, they then successively study the characteristics of figures, the content of areas, and angle sizes on the basis of corresponding worksheets. Two classroom periods each are earmarked for this purpose. In this process, the students have the opportunity of working creatively, learning interrelationships independently, and training their geometric imagination on concrete objects. The geoboard makes it possible to utilize different channels of perception and thus address different types of learner. The students actively use their own hands, view the shapes formed on the board, transfer them to their notebook as drawings, and describe their results in writing. On the basis of solution sheets, they can check their results on their own and thus receive immediate feedback on their work. Instructions for assembly and work and solution sheets (in German language) are available at www.sinus-bayern.de.

In addition, the geoboard can also be used in subsequent grades for experiments designed to teach or illustrate laws on the basis of geometric figures.

2 by Elke Frey, Roland Grebner and Karl-Willi Strobel-Rötter
3 Taken from: Das Zahlenbuch 5, Klett und Balmer, Zug 1999
(from “The Number Book, 5th grade”)

Fig. 3: Geoboard
Interactive Worksheets

The following example illustrates a method for encouraging independent student activity using the computer. Interactive worksheets that enable students to retrieve necessary basic knowledge, various aids, and control results via links are very suitable tools for supporting them in tackling more complex problems on their own.

Example: Problem involving extreme values
Distance of a point from a function graph

With the guidance of 4 HTML worksheets provided at www.sinus-transfer.eu the students independently develop a solution for the problem involving extreme values, i.e. determining the distance of a point from a function graph. At least three class periods have to be planned for this assignment, in which the class works solely on computers. As a prerequisite, the students should be able to derive polynomial functions and calculate extreme values.

Worksheet 1:

Problem involving extreme values

Problem: Distance of a fixed point A to a point on a curve B.

Move point B on the curve.
Where approximately is the point on the curve with the shortest distance to point A?

This sheet serves as an introduction to the subject matter. The students examine the distance of a specified point from a point that can be moved on a parabola. In doing so, they determine experimentally the location of the point on the parabola for which this distance is the minimum.

4. by Thomas Tippelt
Worksheet 2:
In this assignment, the students first examine the distance from the origin to a given straight line on the basis of geometric considerations. To solve the problem, the students then have to identify the functional relationship between the x coordinate of a point on the straight line and its distance from the origin. Via the links “More info 1” and “More info 2” the students can refresh their basic knowledge as needed for the equation of a straight line and to determine the distance between two points in a coordinate system. Furthermore, they have access to aids enabling them to solve the problem on their own. Finally they can check their results independently.

Problem involving extreme values | Page 2

We begin with the distance from a point to a straight line.

Move point B on the straight line g.
1) Form the equation of the straight line.
2) Calculate the distance (i.e. the shortest distance) between point C and the straight line g.
3) Calculate generally the distance from any point B on the straight line g to point C.

Aid 1:
The equation of a straight line has the form y = mx + t.
The easiest way is to determine the slope m and the intercept t from the drawing.

Aid 2:
The distance from a point to a straight line is the length of the perpendicular.
1st option: You have to form the equation of a straight line that is perpendicular to the straight line g and goes through point C. (Note: The slope of a perpendicular to g is the negative reciprocal of the slope of g.) Then calculate the point where these two straight lines intersect. You can calculate the length of segment [CS] using the Pythagorean theorem.
2nd option: You know the lengths of the two legs in the right-angled triangle ACD. Using the tangent function, you can then calculate the angle at A. Now look at the right-angled triangle ACS where S is the point of intersection of the perpendicular from C to g. With the correct angle function you can calculate the length of segment [CS].

Aid 3:
A random point B on the straight line g has the coordinates B (x / 0.5 x + 2). You can again calculate its distance to point C using the Pythagorean theorem.
Worksheet 3:
The functional relationship between the x coordinate of a point on a straight line and its distance \( d \), or the square of this distance \( d^2 \), from the origin, as worked out in worksheet 2, is illustrated more clearly here by developing the graphs for the two relevant functions point by point. After that the distance between the origin and the given straight line is determined analytically. Here again, learning success is supported and ensured via two help links and an opportunity for comparing results.

Worksheet 4:
Here the students transfer the knowledge they already have to the problem of determining the distance from the origin to a parabola. If needed, information on building the vertex form of quadratic functions is made available to them, as are two aids.
Problem involving extreme values

We continue with the distance from a point to a parabola.

Move point B on the parabola p.

1) Form the equation of the parabola.
2) Calculate the shortest distance from point A to parabola p.
3) Calculate generally the distance from any point B on the straight line g to point C.

Aid 1:
The parabola is open to the bottom and wider than a normal parabola. The vertex is at (0 / 4). In addition, the parabola runs through point C (2 / 2). The easiest way is to build the vertex form of the parabola with the help of the above information.

Aid 2:
The coordinates of a random point on the curve are B (x / -0.5 x² + 4). You can again calculate the distance from this point using the Pythagorean theorem. When you move point B, you can see below the graph of the function that assigns the length of the segment [AB] to the x value of point B on the curve. Form the term of the function that assigns the square of its distance to A to the x value of B and discuss it.

This interactive method gives students the opportunity of going back to a simpler situation at any time, in spite of the increasing degree of abstraction, to gain a better understanding of the problem. Furthermore, it is always possible to move a point on the respective graph on the worksheets using the mouse and then read off the measured distance of the given point from the current point on the graph. That means the students can quantitatively check their approaches to finding the solution at any time.
Making Physics a Hands-On Experience

For many students physics is an unpopular subject. This is because they frequently see no connection between physics and everyday life and resent the large number of calculations involved. In the new curricula there are clear indications that novel approaches have been targeted to deal with this problem. The direction now taken leads away from overly “calculational” physics toward practice-oriented natural science.

Some tried and tested experiments and projects that focus on sensory experiences on the part of students (or enable them in the first place) are presented in the following.

Experiencing darkness

In an introduction to optics, the students experience the significance of light and shadow for spatial vision on the basis of an analysis of self-luminous and illuminated objects. This classroom period was based on suggestions from the Humboldt University in Berlin.
**Preparation**
- Several objects made of styrofoam (sphere, cone, cuboid, ...) are placed on a desk, some cardboard discs are hanging on the back wall. In addition, there is a spherical lamp on the table that is connected to the power supply via a dimmer.
- The entire physics room is totally dark, even rays of light under the door or holes in the blackout system interfere with the experiment. The students cannot see the set-up and are unfamiliar with it.

**Procedure**
- First the teacher waits a few minutes so the students can take in the absolute darkness in complete silence. The experience of perceiving the environment without light and solely via other stimuli is extremely impressive. Students can, for example, perceive the teacher walking quietly among them through the slight noise he makes or the heat he gives off.
- Now the setting is illuminated very gradually from the side by means of a lamp and a dimmer. Gradually the objects become visible.
- Then the spherical lamp is turned up very gradually via the dimmer, and the brightness of the outer lamp is reduced.

**Observation**
Depending on the type of illumination, the spherical lamp appears two-dimensional or three-dimensional. If it is self-luminous, it appears like the disc on the rear wall. If it is illuminated, it appears three-dimensional like the styrofoam sphere. Seeing involves both light and shadow.

**Homework**
Describe your feelings and observations. When do the objects appear two-dimensional and when three-dimensional?

The effect experience in the „lab“ can also be repeated impressively in a natural setting by observing the moon with a telescope and the sun with appropriate eye protection.
Law of leverage and the human forearm

Various studies have shown that pointing out interdisciplinary interrelationships between physics and biology or medicine contributes to an increase in motivation, particularly with girls.

The following example illustrates the law of leverage on the student’s own body:

➔ What force do your biceps bring to bear to lift a mass weighing 5 kg?
➔ How long are the lever arms in each case?

Physics turns into a hands-on experience by means of measurements undertaken on the student’s own forearm or on his neighbor’s.

Energy conversion in a dynamo

The jocular saying “We get our electricity out of the wall” represents an everyday experience for many. As a rule, the fact that considerable energy is necessary to operate the numerous electrical appliances in our daily life is something we know but that knowledge remains abstract. The following experiment is a suitable way of making this expenditure of energy perceptible.

While a student turns a dynamo at a constant speed, more and more light bulbs are gradually connected to the circuit in parallel. The increasing expenditure of energy necessary to make the light bulbs light up is clearly noticeable for the person turning the dynamo. The result of the calculation is substantiated by this experience.

Human circuits

The following experiment is both astounding and instructive. Several students form a line and hold each other’s hands. The first and last student each touch the contact of a power cable connected to the poles of a battery (4.5 V are enough!). This creates a closed circuit. A sensitive ammeter is connected to this circuit (μA range).

Both parallel circuits and combined circuits can be set up with chains of people in this way. While a student turns a dynamo at a constant speed, more and more light bulbs are gradually lit.
connected to the circuit in parallel. The increasing expenditure of energy necessary to make the light bulbs light up is clearly noticeable for the person turning the dynamo. The result of the calculation is substantiated by this experience. The ammeter shows how the current changes when the human circuit is interrupted by just a few people in the chain people letting go of their hands.

Density of a person (home experiment)

Home experiments can play a key role in creating a link between everyday experiences and physical phenomena. The “Experience report on the SINUS BLK (Bund-Länder Commission) program in Bavaria” goes into this point in detail.

Here is another example:

Determine the density of your own body using scales and a bathtub.

Resources: bathroom scales, bathtub, 10-liter pail with water-level marker, one helper

Fill the bathtub half full of water and mark the level of the water 1 with a suitable marker or adhesive strip. Get into the tub and submerge your body. Your helper marks the new water level 2. Get out of the tub and using a pail, pour water into the bathtub until water level 2 is reached (measure water volume in liters while doing so!). The volume of water is equal to the volume of your body. Using bathroom scales, you can measure your mass. This means you can calculate “your density”.

A more extensive task could be:

Determine the density of a bread roll.

Other ways of making physics a hands-on experience can be found on the Internet at www.sinus-bayern.de (in German language). They include learning stages on refraction phenomena and suggestions for inventor competitions that are also suitable for project days and school celebrations.
Developing Novel Tasks

To date, tasks have mainly been used for practicing or testing subject knowledge. However, they can and should contribute to the development of personal skills of the kind stipulated by the national educational standards. In developing a task culture, it is important to build on tried and tested practical experience to develop ways of formulating tasks, and methods of working on a task that encourage independent student activity and strengthen skills in taking an insightful, communicative, and evaluative approach to learning. Such tasks can then become genuine “learning tasks” that can either (a) form the core of the lesson, (b) be used at the beginning of a lesson as a starting point for ideas or to continue thought processes that have already been initiated, or (c) to link up established and novel knowledge during practice phases.

by Stefan Grabe and Wolf Kraus
with the assistance of Karl Bögler, Dieter Fiedler, Martin Jochner, Axel Kisters, Claudia Schneider, and Johann Staudinger

1. www.kmk.org/schul/home1.htm
Characteristics of an Advanced Task Culture

Advanced tasks have one or several of the following characteristics:

➔ They help students to take an **active** part in **knowledge acquisition** and to think about the material presented to them in their own way. For example, students may be asked to create new tasks or find missing information on the internet or through other kinds of research.

➔ Their **cumulative character** connects up basic knowledge.

**Example:** “Compare the [previously unknown] hand skeleton of the common pigeon with the [previously discussed] human hand skeleton.”

➔ They contain **open questions** giving students opportunities to develop their own hypotheses and solutions.

**Examples:** Explain the concepts of homology and analogy using an example of your choice.
Design an experiment to find out what colors dogs are able to see.
These are two recipes for red cabbage [...].
Find a hypothesis to explain why the cabbage turns blue in one recipe and red in the other.

➔ This encourages students to use source texts, pictures, cartoons etc., to formulate their own questions.
They refer to a variety of materials in order to practice dealing with different types of presentation format (texts, diagrams, ...) (see task “Catalyst from outer space”, p. 102).

They stand in a contemporary and motivating context (reference to real life).

They require skills in several areas. The following section explains how skills acquired in the fields of subject-related knowledge, more profound comprehension, communication, and evaluation can be included in all this (see also the “mycorrhiza” task on p. 106).

Setting “New” Tasks

First of all, we want to take a brief look at the skills defined in the national educational standards, as they provide important ideas for task variation. The national educational standards set out by the Standing Conference of the Ministers of Education and Cultural Affairs for intermediate school-leaving certificates define four areas for the natural sciences, which are outlined in brief below:

<table>
<thead>
<tr>
<th>Skills</th>
<th>Range of Requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject-related knowledge</td>
<td>Reproducing knowledge</td>
</tr>
<tr>
<td>Improving comprehension</td>
<td>Applying knowledge</td>
</tr>
<tr>
<td>Communication</td>
<td>Transferring and linking knowledge</td>
</tr>
<tr>
<td>Evaluation</td>
<td>Describing subject-related methods</td>
</tr>
<tr>
<td></td>
<td>Using subject-related methods</td>
</tr>
<tr>
<td></td>
<td>Selecting and applying subject-related methods to solve a given problem</td>
</tr>
<tr>
<td></td>
<td>Working with prescribed presentation formats</td>
</tr>
<tr>
<td></td>
<td>Using suitable presentation formats</td>
</tr>
<tr>
<td></td>
<td>Selecting and using presentation formats independently</td>
</tr>
<tr>
<td></td>
<td>Understanding evaluations provided</td>
</tr>
<tr>
<td></td>
<td>Assessing and commenting on evaluations provided</td>
</tr>
<tr>
<td></td>
<td>Making own evaluations</td>
</tr>
</tbody>
</table>

Please note: A detailed table and notes on the different categories are available at [www.sinus-bayern.de](http://www.sinus-bayern.de) (in German language).

Experience has shown that these areas can provide a host of diverse and challenging tasks. The following paragraphs explain this in more detail and provide some examples.
Knowledge of subject-related content has always been an intrinsic part of any lesson (so far in the form of reproduction, reorganisation, transfer, or problem-solving thought processes). Linking the knowledge of facts with the basic concepts described in the subject profiles of the curriculum opens up new avenues to explore.

Intervening at various stages in this process leads to a wide range of potential questions.

➔ What questions can be posed as a result of this observation process?
➔ What hypothesis was the experimenter following up?
➔ Plan an experiment in order to...
➔ What additional conditions would you need to know or keep constant?
➔ What can you deduce from your observations of this experiment?
➔ What additional experiments would be necessary in order to....?
Practicing the systematic switch between various presentation formats is an important basis for enhancing comprehension:

- Explaining and interpreting diagrams, cartoons,...
- Creating diagrams, mind maps, structural diagrams,...
- Deciding autonomously what to put in the exercise book

Suitable teaching methods can also be employed to practice (a) skill in presenting arguments that are tailored to the subject and the addressee(s) and (b) justifying suggested solutions:

- Learning by teaching [Lernen durch Lehren – LDL]

Questions on ambivalent subjects requiring both subject-related and value-related assessments are particularly suitable for this purpose.

Sample task „Catalyst from outer space“

One of the components in the large-scale production of a halogenic plastic is a certain metal-oxide catalyst. However, technical journals have recently been praising a new polymer catalyst that has been developed for this process and have also referred to an interesting meteorite find. This meteorite contains an extremely rare metal, and initial tests using minimal amounts of it have shown that it is a suitable catalyst for the above-mentioned process. In addition, it has been established that the use of these catalysts triggers various side-reactions, depending on the operating temperature. In some cases these side-reactions lead to the production of considerable amounts of highly toxic dioxin. The following table shows data from a comparison of the three catalysts:

---

<table>
<thead>
<tr>
<th>Catalyst</th>
<th>Plastic production output at 200°C</th>
<th>Plastic production output at 400°C</th>
<th>Price of catalyst per kg</th>
<th>By-product dioxin in µg at 200 °C</th>
<th>By-product dioxin in µg at 400 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional metal oxide catalyst</td>
<td>35 %</td>
<td>50 %</td>
<td>270 €</td>
<td>not traceable</td>
<td>&gt; 3</td>
</tr>
<tr>
<td>Newly developed polymer catalyst</td>
<td>37 %</td>
<td>65 %</td>
<td>350 €</td>
<td>&lt; 1</td>
<td>&gt; 2</td>
</tr>
<tr>
<td>Metal catalyst from meteorite</td>
<td>50 %</td>
<td>40 %</td>
<td>–</td>
<td>not traceable</td>
<td>&gt; 2</td>
</tr>
</tbody>
</table>

### Material 2

**Annual CO₂ emissions in large-scale plastics production**

![Graph showing CO₂ emissions](image)

### Question

Assess the suitability of the three catalysts based on the data provided. [*Evaluation III*]

### Expectations

Number of different viewpoints: economic, ecological; number of logical arguments

#### Ecological:

To protect the environment, CO₂ emissions should be kept low: meteorite or new catalyst suitable for low temperatures;

Problem: production of dioxin, meteorite or metal oxide at 200°C, but very little material available for large-scale production (meteorite)

#### Economic:

Excellent yield at 400°C but higher costs, catalyst is not “used up” (polymer catalyst)

The arguments need to be weighted. The meteorite or a metal oxide is more suitable from an ecological point of view, but the new polymer catalyst is more suitable from an economic point of view.
Other suitable subjects include pest control (e.g. DDT against malaria carriers), animal husbandry, genetic engineering, pre-implantation genetic diagnosis, and abortion.

### Additional Strategies for Task Setting

**Use of operators**

Clear work instructions (operators) as described in the uniform examination requirements (EPA)\(^2\) for the Abitur (higher school-leaving certificate) make the tasks clearer and provide ideas for preparing questions.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Description of expected performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis and examination</td>
<td>Identifying essential parts or characteristics in response to a specific question; examinations may also include additional practical parts.</td>
</tr>
<tr>
<td>Appraisal</td>
<td>Creating a correlation between data, individual results, and other elements, and drawing a conclusion if possible</td>
</tr>
<tr>
<td>Assessment</td>
<td>Making and justifying an independent assessment of facts using subject-related knowledge and methods</td>
</tr>
<tr>
<td>Clarification</td>
<td>Illustrating facts and making them understandable by providing additional information</td>
</tr>
<tr>
<td>Checking or examining</td>
<td>Measuring evidence or statements against facts or intrinsic logic and uncovering any contradictions</td>
</tr>
<tr>
<td>Comparing</td>
<td>Identifying common ground, similarities and differences</td>
</tr>
<tr>
<td>Deduction</td>
<td>Drawing reasonable conclusions from essential characteristics</td>
</tr>
<tr>
<td>Description</td>
<td>Reproducing structures, facts, or correlations in a structured and articulate way using subject-related language</td>
</tr>
<tr>
<td>Determination</td>
<td>Finding a correlation or a solution and formulating the result</td>
</tr>
<tr>
<td>Discussion</td>
<td>Comparing arguments and examples relating to a statement or a theory and weighing them against each other</td>
</tr>
<tr>
<td>Developing a hypothesis; synonymous with forming a hypothesis</td>
<td>Formulating well-founded assumptions on the basis of observations, examinations, experiments, or statements</td>
</tr>
<tr>
<td>Drawing</td>
<td>Producing a graphic representation of observed or prescribed structures that is as accurate as possible</td>
</tr>
<tr>
<td>Evaluation</td>
<td>Measuring facts against recognisable value categories or known evaluation criteria</td>
</tr>
</tbody>
</table>

---

\(^2\) From: Uniform examination requirements for the Abitur (higher school-leaving certificate) in the subject of biology (decision of Conference of the Ministers of Education and Cultural Affairs on 1st December 1989 in the version dated 5th February 2004)
Tasks covering the four areas in which skills are to be developed/enhanced can be usefully related to everyday school situations. It will only be possible to use some of these tasks in lessons. This offers a good opportunity for internal differentiation.

Smoking – a Health Hazard
During a school outing (on foot), your class is caught in a heavy rain shower. Fortunately, you soon find a restaurant where you can get shelter and refreshments. As the non-smoking restaurant is too small and there is only one person in the smoking room, your teacher allows some of you to sit there. Your teacher asks the smoker to extinguish his cigarette in no uncertain terms, but the smoker refuses to do so. The next day, the teacher returns to the subject of smoking and carries out the following experiment:

- Second cigarette filter
- Glowing filter cigarette
- Glass tube
- Water jet pump

Taking up everyday situations
Possible tasks

<table>
<thead>
<tr>
<th>Range of Requirements</th>
<th>Skills area</th>
<th>Communication</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Subject-related knowledge</td>
<td>Enhancing comprehension</td>
<td></td>
</tr>
<tr>
<td>Describe the potential consequences for the smoker’s health.</td>
<td>Take adequate notes of the experiment.</td>
<td>In a role play, the person sitting next to you will take on the role of the smoker. Explain the potential health risks of smoking to him/her.</td>
<td>State your teacher’s reasons for asking the smoker to put out his cigarette.</td>
</tr>
<tr>
<td>II</td>
<td></td>
<td></td>
<td>When the smoker offers you a cigarette, the teacher intervenes. Assess your teacher’s behavior.</td>
</tr>
<tr>
<td>Describe the consequences the smoker’s decision may have on your health.</td>
<td>Make an assumption the teacher could test with this experiment.</td>
<td>Once you have explained the potential health risks of smoking, your partner responds that, in adverts, smokers always look the very picture of health. Give your views on this.</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>One student says that, as an athlete, he would rather not be subjected to passive smoking. Explain why many athletes would agree with him.</td>
<td>One cigarette manufacturer claims that when smoking a hookah pipe, in which the smoke is led through water before being breathed in, all the tar contained in cigarettes is absorbed by the water. Devise a simple experiment to check whether this claim is true.</td>
<td>Make a well-founded statement providing your view of the situation described.</td>
</tr>
</tbody>
</table>

Reversal and variation of existing tasks
(e.g. from old tests or school books)

Sample task “mycorrhiza”

Suggestions can be taken from the following task variations on the subject of “mycorrhiza”, which can be assigned to the topic „Basic interrelationships between living organisms“:

Once the subject of “mycorrhiza” had been discussed in class, the following task was set in a written test for year 8 (9-year Gymnasium):
**Task**

The following experiment was carried out in a research laboratory:
1,000 spruce shoots were grown under identical conditions. However, the soil in which 500 of the shoots were grown had been treated with a fungicide before the experiment. These shoots did not grow as well as those in the other group. Explain this observation.

The task could be varied as follows in order to take into account the four central skills targeted in the national educational standards:

**Task variation 1**

The following experiment was carried out in a research laboratory:
The length of 1,000 spruce shoots was measured every two days for two weeks. Prior to the experiment, the soil in which 500 of the shoots were grown had been treated with a fungicide. Otherwise, all 1,000 shoots were grown under the same conditions, i.e. in the same natural woodland soil.
The observations are shown in the graph below.

1. Describe the curve shapes [communication I] and explain the observations [subject-related knowledge II].
2. What questions were the scientists investigating with this experiment?
3. Name the conditions that have to be kept identical for this experiment [enhancing comprehension II].
4. Give your views on the use of fungicides on fields close to forests [evaluation III].
Task variation 2
Task text as above.

Create a labeled diagram for the expected observations [communication III] and give reasons for the curve shapes [subject-related knowledge II].

Task variation 3

Scientists want to investigate whether mycorrhiza also has an effect on the growth of spruce shoots.

Devise a suitable experiment to investigate this question. [enhancing comprehension III].

Expectations: blind test, use of fungicide, just one variable, suitable duration of experiment, possibly larger number of samples

Other possible variants
➔ Assign a key [communication II];
➔ One graph provided, a second to be added [communication II];
➔ Discuss advantages of greater number of samples [enhancing comprehension I];
➔ How does the growth of shoots take place?
What contribution does mycorrhiza make? [subject-related knowledge I]

Source texts, illustrations, diagrams from school or reference books, magazines or newspapers may be used as starting material.

Sample task “mineral water”
Tasks can be created using source texts such as the article „Mineral water – from the source to the bottle“, which was published in the „Spektrum der Wissenschaft“ magazine in August 2003. Some of the following tasks do not directly refer to the material used. The text is rather to be seen as a source of inspiration.
Possible tasks

1. Compile the list of contents of a mineral water of your choice, using a suitable diagram of your own choice [C II].

*In mg/l: Cations: sodium 36.9; potassium 4.6; calcium 272.4; magnesium 71.3; manganese <0.05; anions: fluoride <0.3; chloride 95.7; nitrite <0.02; nitrate <2; sulphate 597.6; hydrogen carbonate 390.*

2. There are three different mineral waters available. Based on the list of contents, discuss which mineral water is best in terms of quality [E II].

3. The following experiment was carried out: One mineral water is heated to boiling point in a flask. The gas discharged is conducted into lime water through a glass tube. The lime water becomes cloudy when the gases enter it.

![Diagram of mineral water and lime water](image)

What could the experimenter have been trying to find out? [S I/D II]

The above experiment is to be used to compare four other mineral waters. What factors must be kept constant? [D II]

What inquiries might be connected with this? [D II]

Creating problems by giving too much (unnecessary) information or leaving out necessary information (over- or underdefined tasks). A task related to this and more examples are available at [www.sinus-bayern.de](http://www.sinus-bayern.de) (in German language).
Using Tasks in Designing Maths Lessons

A brochure published in 2002\(^1\) discusses the different aspects of advancing the “task culture.” The previous chapter of this publication has investigated the role educational standards can play in this context, with examples from biology and chemistry classes. The essential ideas here can be transferred to other subjects, and to mathematics in particular. Accordingly, this section discusses some additional aspects.

The Role of Tasks in Mathematics Lessons

Tasks play an essential role in math lessons. More than in other school subjects, they are essential for illustrating abstract correlations, practicing algorithms, and ultimately verifying student progress. In short, tasks are a way of both illustrating and practicing reproducible skills and are also used in tests.

But in these remarks one essential aspect is neglected, an aspect that can make tasks appealing to both dyed-in-the-wool mathematicians and beginners alike: the challenge of regarding mathematics from a “transactional” viewpoint (P. Gallin). This is a key feature in heightening the subject’s appeal.

The right kind of tasks can trigger personal engagement and independent exploration. For this to happen, interesting problems must be placed at the heart centre of the lesson, and personal involvement must be encouraged before the theoretical side is explained. One didactic concept of this kind is Ruf/Gallin’s\(^2\) “dialogue-oriented” learning summed up by the triad “I – you – we”. In a first step, the students analyze a task on their own („task”, see below) (”I” aspect). Then they enter into dialogue with the “you” (teacher, fellow student), who will furnish constructive feed-back. This step has
nothing to do with looking for errors. It is about reinforcing useful or original approaches. Finally, the regularized mathematical solution is established for everyone ("we"). This "I – you – we" principle has been adopted by many teachers, even those reluctant to comply with the methodological suggestions made by Gallin/Ruf. Students who have already analyzed a technical task more or less successfully are much more interested in finding out the solution than if they had simply reproduced it, and this fact alone is a strong argument in favor of this approach. Someone who has started thinking about a problem will want to know "how it works out."
The four examples that follow illustrate the approach in different ways. They can be used to help students get started, and they also contain demanding challenges. The first three tasks mainly serve to introduce and develop a new topic, while the fourth example is an example that can be used for practice periods.

Examples

The first example presents incentive material with which students in year 8 or 9 can be confronted with the subject of gradients before the concept itself has been introduced.

Downhill Martian
You have to be a little crazy to take part in the XSpeed Championship in Verbier, Switzerland. The skis are insanely long (2.5 metres), the slope is insanely steep (up to 90 percent gradient), and you have to wear this insane suit. The skin-tight latex suit and the aerodynamic helmet are designed to minimize air resistance. Speed skiers travel insanely fast – Italian Simone Origone (photo: dpa) holds the world record, 251.21 kilometers per hour, and Swiss skier Philipp May also topped the 250 kph mark last year (...).
This newspaper clipping from the *Süddeutsche Zeitung* dated 21st April 2007 provides a good start with the opportunity to compare the photo with the statement „up to 90% gradient.“ This automatically leads to possible tasks. Physics tasks on the same subject (in German language) can be found at [www.leifiphysik.de](http://www.leifiphysik.de) (year 11 (9-year Gymnasium) under “Musteraufgaben” (sample tasks). The introduction of new subjects based on concrete examples is often recommendable and can easily be done by working on interesting tasks beforehand (including some taken from the textbook).

**Tasks focusing on argumentation and communication**

Identical cubes are arranged in such a way as to look like image 1 from the side and like image 2 from the front.

![image 1](image1.png) ![image 2](image2.png)

With 20 cubes this is what the oblique view looks like.

![3D view](3d_view.png)

a) Could the above images be achieved with fewer cubes?
b) Is there a minimum number?

The surprising solution to this task is that a mere 6 cubes are sufficient. As the unnecessary cubes are removed one by one, a reason must be given for each step. Finally, the supposition that fewer than 6 cubes would not be enough is proven as soon as you look at the structure from above and realize that in each line and each column of the 4x4 pattern of the bottom layer there is exactly one cube.

*Based on de Lange, J., Utrecht: personal communication*
An animated version of the process can be viewed at www.sinus-bayern.de. For this task, personal engagement and communication with partners are important. Access to the problem is motivating and action-oriented, and it is a good example of a demanding task that enhances spatial imagination. In addition, it calls for a critical examination of representations with missing information.

Differentiating tasks encouraging meaningful mathematical activity at any level are a suitable way of challenging and activating all students. A well-formulated task must contain three aspects:

The introduction should lead into the problem and not pose an insurmountable hurdle for anyone. Of course, at its center there has to be a substantial mathematical problem relating to the current curriculum – the heart of the matter. The third aspect of a differentiating task motivates students to rise to intellectual heights that not everyone will always reach. This “ramp” can create a link, facilitate discoveries, and call for new problem-solving strategies. The ramp may also contain a generalization of the special case that lies at the core of the task.

The following example starts off with simple fractions, which can be made more difficult by choosing more demanding numbers. The heart of the matter is a counting problem:

### Introduction
The numbers 1, 2, and 3 are given. Form all possible fractions from any two of these numbers, and sort them according to size. (Numbers may be used several times!)

### Heart of the matter
How many possibilities are there using numbers 1, 2, 3, 5, 7?
How many of these fractions are smaller than 1?

### Ramp
How many possibilities are there using numbers 1, 2, 3, 4, 5, 6,…..n ? Are all the fractions different from one another? Explore further!
In class, the teacher should first ensure that all the students have properly understood the task and know what they are focussing on. In this case, that means forming fractions with one-digit numerators and one-digit denominators from the numbers given, and using these numbers several times as necessary.

A table soon makes it obvious that the number of possibilities is the same as the square of the number of available figures.

The question about the number of possible fractions smaller than 1 can then be easily answered directly or – without using concrete numbers – by illustrating the problem using tiles.

This result also contains the solution to the famous Gaussian problem, where the total of natural numbers smaller than or equal to n—1 (in the counting method used here) must be calculated. An extensive presentation of this very graphic proof for the Gaussian formula and suggestions for „ramps“ can be found at www.sinus-bayern.de (in German language).
Undoubtedly, frequent practice and repetition of routines play an essential role in task-solving both in class and at home. Our intention in this section is to demonstrate how suitable tasks can lead to long-term success through exercises that go well beyond the “blind” rehearsal of algorithms that have not been understood. Teachers frequently complain that their students use inadequate solution methods. But it is rare for activities in class to focus on this problem. Here is an example. Students sometimes solve the equation $3x^2 = 48$ by using the formula or by completing the square. Problems such as: $x(x + 2) = 0$ can be solved in the same way after multiplication. When it comes to series of problems, this can be counteracted by not just presenting the problem as it is formulated in the book:

**Calculate the roots of the function**

<table>
<thead>
<tr>
<th>a) $x^2 + 8 = 0$</th>
<th>b) $(2x + 2)x = 2$</th>
<th>c) $x^2 + x = 2x - 2$</th>
<th>d) $x^2 - 13x = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>e) $-x^2 + 3x = x^2$</td>
<td>f) $2 + x + x^2 = 2$</td>
<td>g) $x^2 + 6x + 5 = 0$</td>
<td>h) $x(x + 2) = 0$</td>
</tr>
<tr>
<td>i) $x^2 + 2 - 15x = 0$</td>
<td>j) $2x^2 - 5 = 2x + 5$</td>
<td>k) $x = x^2$</td>
<td>l) $x + 2 + x^2 + 2 = 0$</td>
</tr>
<tr>
<td>m) $x(x + 2) = 0$</td>
<td>n) $x(x + 2) = 2$</td>
<td>o) $2x^2 + 2(x - 2) + 3 = 0$</td>
<td>p) $2(x + 2)(x - 2) = 0$</td>
</tr>
</tbody>
</table>

Some inspiring questions:

➔ Form groups and explain your decision.
➔ Which tasks can you solve and which ones can’t you? Why?
➔ Which tasks could be simplified by minor conversions?

Such questions direct attention to the various problems and the conscious choice of solution procedures. They are “differentiating” because they can be approached at different levels. Some students will need to work through a number of problems, others will easily recognise the structures behind it all. Such questions are always independent of the current topic being worked on in class and it makes good sense to pose them in any context.

---

4. According to Leuders, T.: Reflecting practice even with “forests” of tasks in MNU 59/5 Verlag Klaus Seeberger Neuss, 2006
Experience

Outcomes of Process Evaluation

Introduction

The SINUS Transfer program was evaluated between 2004 and 2007 whilst the process was ongoing. Each year, surveys were carried out to ascertain the participants’ response in terms of motivation for participation, relevance for lessons, satisfaction with the work of the coordinators, development of lessons, and cooperation. The results from the three surveys in 2004 – 2006 display showed only slight differences between SINUS twosomes and between school types. For this reason we have used the mean value for the entire sample in this brochure. Nevertheless, these slight differences are an important survey result, since they show that – irrespective of the personal touch provided by the individual twosome – the approach has been successful in all three school types.
Relevance and Productivity of Training Seminars

The following diagram gives an overview of the general assessment relating to the relevance and productivity of the training seminars that form the core of the SINUS Transfer program. Participants were asked to answer the following questions:

➔ Do the teachers taking part in the program make use of material from other teachers at their school (or another school) in their lessons?
➔ Are the contributions of other participants during the training seminar useful, are the sessions always effective, are there high-level technical discussions, do they contribute to achieving the objective of the seminar?
➔ There were additional questions regarding mutual inspiration and whether or not individual participants dominated a seminar.

The overall mean value reflects a positive assessment of training seminars with respect to the questions asked.
Satisfaction with the Work of Coordinators

The acceptance survey focused in particular on satisfaction with the work of the coordinators. The questions covered the following aspects:

➔ Do the coordinators provide stimulus prompting teachers to rethink and change their teaching approach?
➔ Do they show an interest in the status of the work within the working groups and in any problems and difficulties cropping up?
➔ Do coordinators allow scope for own ideas and work on the modules, do they give constructive feedback, are they friendly, and do they try to create a relaxed atmosphere during the training seminar?
➔ Do they give the impression of having a personal interest in improving mathematics and science teaching, and do they provide helpful support for structured work?

The results of the three surveys reflect very high overall mean values in terms of satisfaction with the coordinators.

Figure 3: Satisfaction with the work of coordinators
Scale from 1 (“low satisfaction”) to 4 (“high satisfaction”)
Perceived Developments and Positive Aspects

The evaluation of the survey results summarize the teachers’ assessments concerning changes to their teaching approach that have already materialized. They contain factors such as the opportunity for innovation in lessons or a positive attitude manifested by a spirit of optimism, enjoyment, and confidence in trying out new approaches in class.

The scores are essentially positive but also reveal that changing one’s teaching approach is a process that takes considerable time.

Cooperation Inside and Outside School

Finally, the survey asked about forms of cooperation, such as exchanging teaching materials and tests or cooperation in the further development of teaching approaches.
Once again, the mean values of the total sample are rather low. Regular cooperation among teachers does not yet appear to have become routine. A detailed analysis shows that there are distinct differences between schools with regard to cooperation.

Conclusion

The results of the evaluation allow positive conclusions to be drawn regarding the success of the SINUS Transfer strategy for the further training of teachers. Beyond the original program objective – the transfer of results from the model test program SINUS – we have found that continuous support and orientation toward modules can trigger long-term processes among teachers. Work with the professional community and the group of schools provided concrete support directly related to teaching practice.
Teachers’ Experience

As a further assessment of the long-term effects of the SINUS Transfer program alongside the evaluation carried out during the process (see Outcomes of Process Evaluation), interviews with teachers were carried out at two schools more than one year after their participation in the SINUS Transfer program. Here is some selected feedback from these teachers.

Cooperation between Teachers

Cooperation within the professional community or between staff after the end of SINUS seminars received a particularly positive assessment. “A very beneficial factor was the increase in cooperation, or let’s say the exchange of information among staff, as a result of these seminars. We talked a lot more about what each one of us was doing in class.” This helped reduce insecurity and improve professional self-esteem.

Advancement of Teaching Methods

The SINUS Transfer program supported teachers in making progressive changes to their teaching methods. One teacher said: “I already have quite a bit of experience, and I can now add new ideas to this experience. That doesn’t mean that I would suddenly do everything completely differently from one year to the next. It’s a selective thing. I pick out something I want to try out in a particular situation – this could be a new form of lesson, it could be a different task – and apply it where I think fit.”
One teacher describes the changes to his teaching methods as follows: “One thing I do more and more often is to hold back during the lesson, during exercise phases, for example. In the past I thought I had to show the students how to calculate problems as much as possible. Now I have radically changed this. I just show the students how to calculate one problem and then let them try it out for themselves to see what they can do.”

Some teachers reported using activating teaching methods more often than before in order to promote independent study and to convey learning strategies. “That is the point that fascinated me: developing problem-solving strategies and establishing this process in the students’ minds, or teaching them to think more independently.”

The teachers largely agreed that changes to the working attitude and learning behavior of students have to be introduced at an early stage. “That is probably why we worked a lot with students in years five and six, because from the outset you have to explain what group work is and how it works. That makes it easier when the students are in years eight or nine, because by then they know what a homework template is, for example.”
Experiences of a Twosome at a *Hauptschule* (Basic Secondary School)

Exchange at Various Levels

In addition to daily lessons in class, the SINUS Transfer program introduced further training closely related to the instructor’s own teaching and thus to the students who had been getting a very bad press. The idea was to pass onto the teachers the many positive individual experiences and discoveries from other SINUS colleagues, so as to support their daily teaching work.

Exchange at various levels was one of the main aspects here. As we teach classes for the same age group we are able to work together preparing lessons and reflecting on results. Close contact with the teachers we support and the other SINUS twosomes had a positive effect on our own work. In addition, we received a lot of helpful tips from various experts on teaching methodology for individual subjects.

We first of all started with open tasks and tasks from the newspaper. The students managed to convince us that even those who are less gifted are able to develop questions and possible solutions independently. In our enthusiasm about the students’ motivation and stamina we then changed the lessons so that they could work independently on new subject matter in these introductory lessons. This approach gives the teacher time to help weaker students, to understand their various thought processes, and to talk to them.

During the lesson, the teacher displays a lot more patience and calm, and no longer concentrates his/her observations on student deficits. Errors occurring during the learning process are a starting point for subsequent deliberations and discussions with a partner or in groups.

In addition to oral verbalization, writing down thoughts and prospective solutions ideas became more and more important, as did the students’ descriptions of their own progress. The students showed that they were able to assume responsibility for what they learn.
We started off by supporting individual schools in Lower Bavaria, subsequently – due to high demand – several groups of schools. Once teachers had gained their first positive experiences with open tasks, cumulative learning techniques, and independent studying, initial scepticism soon dissolved. We attached major importance to getting colleagues to develop their own materials and methods. The results were presented at the meetings, and experiences were discussed. Different schools tended to focus on different things.

Particular highlights were the training events with distinguished speakers such as Prof. Wilfried Herget and Prof. Gregor Wieland. Many interested and enthusiastic teachers from all over Lower Bavaria came to the events entitled “A Somewhat Different Task” and “Terms and Variables”.

“Learning for Life” – Parent Feedback

The parents interviewed felt that mathematics lessons had become easier to understand for the students. Previously, their children had needed a lot of support with their homework and revision for tests, but now they were progressively becoming more independent, displaying greater initiative and creativity. The parents felt that the success of working with SINUS was not just limited to mathematics, but also noticed positive effects on learning in general. Students stop to think before employing just any formula, analyze tasks more thoroughly, think about problems, and are more ambitious than they were before. On the whole, they are coping better with all their tasks and are learning for life. The parents stressed that their children enjoyed mathematics lessons because the lessons were interesting and varied. This prompted them to try harder and achieve better grades. The pressure and stress formerly associated with tests in mathematics is decreasing because the children are able to mobilize their basic skills/knowledge as they need them.
Feedback from Former Students

At the beginning of the interview, all participants stressed that they had fond memories of mathematics lessons in their last few years at school. They particularly remembered tasks from the newspaper and the pictures associated with them. These pictures helped the students to develop mathematical questions and various approaches for solutions. The students enjoyed both presenting the ideas for solutions developed by their group and the subsequent discussions on the plausibility of the results. Everyone was able to contribute personal skills, and this enhanced motivation and dedication. This type of task also taught the students how to think about a problem for an extended period and to develop their ideas for solving problems instead of using a prescribed formula. During their vocational training, the young people still felt the benefits from this kind of maths lesson, even though they predominantly used formulae at vocational college. When faced with tasks that are not obviously associated with a particular formula, they adopt an independent and creative approach, whereas other students give up quickly if they cannot immediately find the right formula. The ability to calculate without formulae and calculators is indispensable at the workplace and during recruitment tests. Here the daily mental arithmetic tasks created independently by the students have a positive effect.
Outlook: SINUS BAVARIA

SINUS Bavaria

The extremely positive experiences outlined above have led to a further training program being offered for all Bavarian secondary schools as of the school year 2007/08. This program is called SINUS Bavaria and builds on the tried and tested SINUS approach, extending it to encompass new content.

Content

The SINUS Transfer strategy has three focal points:

➔ The advancement of task culture
➔ Independent learning
➔ Securing basic skills through cumulative learning methods
These elements will continue to play an essential role. At the same time, there are a host of other important subjects that should be added to the further training program or receive more attention within it. These include:

➔ Personal support and differentiation
➔ Learning from mistakes
➔ Scientific study methods
➔ Learning how to solve problems

Developments concerning a particular type of school, such as the qualifizierender Hauptschulabschluss (General Certificate of Secondary Education), stochastics at Realschule (lower secondary schools), or lessons that go into greater depth at Gymnasien can also be covered.

SINUS Bavaria serves to promote skills-oriented teaching in line with the national educational standards.¹

 Organisation

One new aspect is that in addition to the regular participation of professional communities or school staff groups, SINUS Bavaria will also invite participation by individual teachers and organize individual events (e.g. professional meetings, teaching conferences). The focus will remain on the subject of mathematics. As before, teachers at Realschule can also participate in physics, while there is an additional extensive program for subjects related to nature and technology, biology and chemistry at Gymnasien.

Experience with the SINUS Transfer program has shown that it can trigger sustainable processes amongst teachers that benefit the quality of teaching. The implementation of similar strategies for the further training of teachers should also be considered for other subjects.

¹ www.kmk.org/schul/home1.htm
www.sinus-transfer.eu

The Federal Ministry of Education and Research (bmbf) has financially supported the English translation of this document.

The translation has been arranged and coordinated by the Chair of Mathematics and Mathematics Education at Bayreuth University.